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Expectation and Variance of Item Resemblance Distributions in a Convolution–Correlation Model of Distributed Memory

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The present paper extends the applicability of distributed memory models that use convolution and correlation as their encoding and retrieval operations by providing analytic derivations of the expressions needed to apply such models to a wide range of experimental paradigms. These extensions allow such models to predict recall and recognition performance for single items and double and triple associations under optimal storage and retrieval conditions as well as in situations of degraded probe information at time of test or incomplete encoding of information at time of study. Part I gives a brief review of one convolution-correlation memory model, TODAM (Murdock (1982). *Psychological Review*, **89**, 609–626; (1983), **90**, 316–338). Part II provides the derivations for expectations and variances of the distributions of dot-product resemblance between an extensive list of memory vectors and probe vectors. Expressions of the first two moments of such resemblance distributions in terms of the model parameters are needed to derive model predictions for such dependent measures as recall or recognition accuracy or d'. Part III illustrates the use these resemblance distribution moments by deriving predictions for the data of three experimental paradigms: (1) recall with partial cues (Tulving & Watkins (1973). *American Journal of Psychology*, **86**, 739–748), (2) recognition of rapidly presented information (Loftus, (1974). *Memory and Cognition*, **2**, 545–548), and (3) recognition in context of a triple association learning task (Clark & Shiffrin (1987). *Memory and Cognition*, **15**, 367–378). © 1988 Academic Press, Inc.

In recent years distributed processing and memory models have attracted much attention by exhibiting such desirable characteristics as direct access, content addressability, redintegrative ability, and soft-fail performance (Gabor, 1969; Hinton & Anderson, 1981; Hopfield, 1982; Rumelhart, McClelland *et al.*, 1986). The distributed storage/processing hypothesis has been embraced by cognitive scientists for its economy in modeling a wide range of phenomena with a single set of rules as well as for its power to model important cognitive functions (such as content addressability) outside the range of other types of models. In modeling human memory, for example, a small set of distributed storage and processing assumptions

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accounts not only for recall and recognition data but also for such "higher" mental functions as categorization (Knapp & Anderson, 1984), probability learning (Anderson, Silverstein, Ritz, & Jones, 1977), and similarity effects (Eich, 1982, 1985). In fact, these abilities turn out to be expected by-products of the distributed encoding and storage mechanisms.

Distributed memory modeling, of course, refers to a class of models, each one differing in its interpretation of the distributed storage and processing assumption as well as in other details. Prompted by the power and properties of optical holographic processes (Gabor, 1949, 1951), a number of authors in the sixties and seventies drew analogies between holograms and the brain viewed as a distributed memory store (Gabor, 1968, 1969; Longuet-Higgins, 1968; van Heerden, 1963) and produced holographic brain and memory models (Cavanagh, 1976; Pribam, 1966; Westlake, 1970). While these models employ the mathematics of the holography principle, namely convolution and correlation, simplifications and modifications of the mathematical representation of encoding and retrieval operations were also suggested. (See Willshaw, 1981, for a review of this literature.)

In the memory domain, the two classes of distributed processing models that have been developed in sufficient detail to make predictions for particular empirical observations mirror this distinction (represented by J. A. Anderson (1973) and collaborators (Anderson *et al.*, 1977) and Pike (1984) one the one hand, and Murdock (1982, 1983) and Eich (1982) on the other hand). These two types of models make similar assumptions about the way information is represented, and about the type of information that is processed. (Information is represented as a feature vector. The type of information processed includes item, associative, and serial-order information.) The two models also agree that the format of the memory store is distributed rather than localized. The models differ in the nature of the storage and retrieval operations and in the resulting particular format of the distributed memory store. Anderson (1973) assumes, for example, that association of two items should be represented by the outer product of the two item vectors (i.e., $\mathbf{fg}'$), resulting in a *matrix* as the memory trace $\mathbf{M}$ of this association. Associative retrieval is modeled by premultiplying the memory matrix $\mathbf{M}$ by probe vector $\mathbf{f}$ (i.e., $\mathbf{f}'\mathbf{M}$) which produces the output $\mathbf{g}'$ (i.e., a vector resembling $\mathbf{g}$). On the other hand, Murdock (1982) in his Theory Of Distributed Associative Memory (TODAM) uses the convolution operation to represent the association of two items (i.e., $\mathbf{f} * \mathbf{g}$), resulting in a memory *vector* $\mathbf{M}$. Retrieval in TODAM is modeled by correlating the probe vector with the memory vector (i.e., $\mathbf{f} \# \mathbf{M}$), which also produces an output vector that resembles $\mathbf{g}$, i.e., $\mathbf{g}'$.

Pike (1984) argues for the superiority of the matrix memory representation over the convolution–correlation vector representation on a number of grounds (neural plausibility, simplicity, and economy, among others). Murdock (1985) countered most of Pike's a priori criticisms of the convolution–correlation approach (a priori arguments can be made for and against either of the two models) and suggested letting the models' ability to account for experimental data be the arbiter in choosing between them.

In fitting the two models to experimental data, Pike is correct in claiming greater simplicity for the matrix approach. Explicit expressions necessary to compute the models' predictions for such dependent variables as recognition accuracy are easier to obtain in the matrix system. The convolution–correlation system however is far from intractable as witnessed by the derivations provided by Murdock (1982, 1985) for the set of experimental tasks which are the domain of those papers. The present paper extends the range of convolution–correlation models like TODAM by providing analytic derivations of the expressions needed to apply such models to a much wider range of experimental paradigms. It is hoped that this will encourage and facilitate the comparison of the convolution–correlation approach with other memory models, be they distributed, search-type, or network-type models. Part I consists of a brief review of TODAM to provide a common language and motivation for the expositions that follow. Part II provides the results of the derivations for expectations and variances of the distributions of dot-product resemblance between memory vectors and probe vectors for an extensive list of experimental paradigms. Expressions of the first two moments of such resemblance distributions in terms of the model parameters are necessary to derive model predictions for such quantities as recall or recognition accuracy or the sensitivity measure d'. As it turns out, determination of the expectations of these resemblance distributions is quite easy. Determination of the variances, which are the variances of sums of components, is more demanding.

Appendix A lists these variances in their extensive form, i.e., each variance as the sum of its component variances and all nonzero covariances. These expressions are useful because of their generality, i.e., they will make it easy for other researchers to develop expressions for the moments of item resemblance distributions when making assumptions different from the ones employed by TODAM (e.g., different distributional assumptions about the component features, or the use of correlated rather than independent random vectors to model item similarity). A different set of assumptions will result in different values for the component expectations, variances, and covariances listed in Appendix B. Once these component expressions have been worked out (a relatively simple task), they can be substituted directly into the formulae provided in Appendix A.

Part III illustrates the use of these resemblance distribution moments to derive model predictions for three experimental paradigms. The first one is cued recall with a cue that is a fragment of the item to be recalled, with data from a study reported by Tulving and Watkins (1973). The second paradigm is recognition of rapidly presented (and thus incompletely encoded) item information, as exemplified in a study by Loftus (1974). The final paradigm to be modeled is the recognition of single items, pairs of items, or triplets of items after a triple association learning task. Data for this paradigm come from Clark and Shiffrin (1987).

PART I: REVIEW OF THE MODEL

Following J. A. Anderson, TODAM represents information (items or events) as a vector of component attributes. The assumptions of Anderson's (1973) matched filter model stem from attempts to provide a plausible neural information processing model, and item or event vectors are represented as vectors of the firing frequency ("deflection, plus or minus, from the neurons' resting level" [Anderson, 1969]) of neurons. Similarly, the features of TODAM should be thought of more as quasi "neural" micro features than as aspects of an item or event that would be identified as "features" by a human observer (even though TODAM makes no attempt at being a neural model, but rather is an abstract mathematical model). The *nature* of these attributes or micro features is not addressed by the model aside from their mathematical representation.

For mathematical convenience, TODAM represents information as *doubly infinite* vectors of micro features with a finite number of N features defined as below (where N is odd) centered on the zero index. These N central features are flanked by micro features which are identically zero for all $i < -(N-1)/2$ and $i > (N-1)/2$. An item vector can thus be represented as $\mathbf{f} = (..., 0, 0, f_{-(N-1)/2}, ..., f_{-1}, f_0, f_{+1}, ..., f_{(N-1)/2}, 0, 0, ...)$. The use of doubly infinite vectors ensures that all vectors (i.e., vectors with a different number of nonzero central elements, or the convolution or correlation of two vectors) have the same number of elements. In practice, this representation amounts to appending as many zeros as necessary to the finite number of nonzero central elements of any vector to perform convolution, correlation, or dot-product computations. The use of an odd number of nonzero features, N, is a notational convenience because it avoids fractional indices and a mathematical convenience because the symmetry of vectors with an odd number of nonzero elements centered around the zero-index element. (It also follows the notation employed in work on convolution and correlation algebras (e.g., Borsellino & Poggio, 1973; Schonemann, 1987), the results of which thus become directly applicable.)

TODAM employs as a model parameter the *number* of nonzero central micro features necessary to represent an item. When TODAM is fitted to model human recall or recognition in a variety of situations, the estimate of the number of nonzero micro features used to represent, for example, a common English word is somewhere in the range of 200 to 500. When one thinks of these micro features as quasi neural, numbers in this range are not unreasonable. Another assumption (made for simplification and prima facie plausible) is that for a given type of stimulus material (e.g., letters, words, pictures) the same number of micro features is used to represent different members of the set. TODAM currently allows for different estimates of N, the number of nonzero central micro features, to represent different *types* of stimuli (e.g., pictures vs words). Again, if one thinks of the number of micro features as reflecting in some way the amount of neural activity during processing, it is not unreasonable to assume a similar amount of activity for all common English nouns, for example.

TODAM assumes that the N central nonzero micro features of a vector are independent random variables that have independent and identical normal distributions with mean zero and variance P/N (for a definition of P see the next paragraph). These assumptions follow those of J. A. Anderson's matched filter model (1973), justification for which is found in earlier Anderson papers (1969, 1970, and 1972). The assumption that the mean value of a micro feature across the set of possible vectors is zero reflects Anderson's interpretation of the elements or micro features of a vector representing the firing frequency of neurons as "deflection, plus or minus, from the neurons' resting level" (1969). It is also made by TODAM because it gives "nearly optimal properties (optimal in the sense that the signal power to noise power ratio is maximized) to the retrieval system" (Anderson, 1972) and because it greatly simplifies calculations. The assumption that all micro features are identically distributed across the set of possible vectors is not only a computational convenience but also the most plausible assumption in the absence of information to the contrary.

TODAM defines P as the average "power" or norm of all item vectors $\mathbf{f}$ across the set of possible vectors (i.e., as the expected value of the random variable resulting from the dot-product operation $\mathbf{f} \cdot \mathbf{f}$ between the two random vectors). According to Anderson (1972), "P in some intuitive sense stands for the 'importance' of a trace ... when traces add together in storage." The constraint of equal power, P, placed on the set of allowable "traces" (i.e., vectors) in his model is a simplifying convenience that Anderson shows to be innocuous in the sense that the results remain the same when P, as in TODAM, is defined as the "'average' power of a trace in the set of allowable traces" (1972).

Parameters estimated or values calculated in Anderson's model as in TODAM are always for "'averages', calculated over many sets of allowable traces" since nothing is known about the "details of any particular trace" (Anderson, 1972). Given the additional simplifying assumption that different traces/items are uncorrelated, Anderson (1972) invokes the Central Limit Theorem to predict that the "value of the sum of many uncorrelated traces approximates a normally distributed random variable," where the summing comes about by the superposition of vectors as in TODAM. In the absence of any information about the distribution of the value of elements of individual traces, one distributional assumption is as good as any, and the assumption that the individual item or trace elements are normally distributed is computationally convenient. In order to ensure that, on average, traces will have equal power, P, and given the additional simplifying assumption that "on the average over sets of allowable traces the statistics of every element be the same" (1972), individual elements are approximated as independent and identical random variables of mean zero and variance P/N. P, in theory, is a model parameter, but for simplicity is usually set to $P = 1$ in TODAM, making item vectors, on average, unit length. To postulate an inverse relationship between the variance of the feature distributions and N, the number of features, prevents the memory system from becoming arbitrarily good by just adding more features.

Formation of an association between two items of length N_1 and N_2 is represen-

ted by the convolution of the two item vectors. For vectors **f** and **g**, their convolution is defined as

$$(\mathbf{f} * \mathbf{g})_x = \sum_i f(i)\, g(x-i) = \sum_i f(x-i)\, g(i) \tag{1}$$

for element x of the convolution, where x and i range from $-\infty$ to $+\infty$, and the elements of the convolution vector take nonzero values for $-([N_1+N_2]/2-1) < x < +([N_1+N_2]/2-1)$. While each element of the convolution is only a single value, that value is a sum of one or more cross products. (In deriving expressions for the variance, one must consider these separate components.)

As discussed below, associative retrieval is represented by the correlation operation, defined as

$$(\mathbf{f} \# \mathbf{g})_x = \sum_i f(i)\, g(x+i) = \sum_i f(x+i)\, g(i) \tag{2}$$

for element x of the correlation, where x and i have the same range as for the convolution.

A graphic illustration of the convolution and correlation operations can be found in Eich (1982). A general reference is Bracewell (1978). Properties of the two operations useful to memory models are described in Murdock (1979, 1982). Tulving (1983) suggests correlation as the retrieval operation highly compatible with a wide range of experimental results. Murdock (1987b) shows that if correlation is the appropriate retrieval operation, then convolution is the associative operation that produces the optimal memory vector.

Higher-order associations such as successive associations or a triple association of three items or events are easily handled by the convolution operation. Since the operation is associative, $\mathbf{f} * (\mathbf{g} * \mathbf{h}) = (\mathbf{f} * \mathbf{g}) * \mathbf{h} = \mathbf{f} * \mathbf{g} * \mathbf{h}$.

A common memory vector **M** is assumed to store item information as well as associative information, the latter subsuming serial-order as well as auto-associative information. (See Murdock (1983, 1987a) for details on serial-order encoding which is modeled by overlapping pairwise associations between neighboring items. See Eich (1985) for details on auto-association, i.e, $\mathbf{f} * \mathbf{f}$, as the encoding operation that could underlie recognition.)

For item information, each item vector is simply added to the memory vector. For associative information, the convolution of the item vectors is added to the memory vector. To avoid saturation of **M** and to model forgetting, the current vector **M** is discounted by a constant $\alpha < 1$, the forgetting parameter, every time new information is added. Other model parameters are the relative weights given to different types of information in a particular situation, where the weights sum to one to model finite attention or processing capacity. To store the jth pair of items presented in a paired associate learning task, for example, the memory vector would be updated as follows,

$$\mathbf{M}_j = \alpha \mathbf{M}_{j-1} + \gamma_1 \mathbf{f}_j + \gamma_2 \mathbf{g}_j + \omega(\mathbf{f}_j * \mathbf{g}_j), \tag{3}$$

where $\gamma_1 + \gamma_2 + \omega = 1$.

In another example, the storage equation for the jth item in a serial-order memory task is

$$\mathbf{M}_j = \alpha \mathbf{M}_{j-1} + \gamma \mathbf{f}_j + \omega(\mathbf{f}_j * \mathbf{f}_{j-1}), \tag{4}$$

where $\gamma + \omega = 1$.

TODAM provides for three types of retrieval. The first is recognition, the process by which a present item is compared with information in memory, resulting in a yes-no response. The second is recall, the process by which an item is generated in response to a question. Lastly, redintegration is the process that reconstructs a complete item from a partial list of its features.

Recognition is modeled by the vector dot product of probe item **f** and memory vector **M** (i.e., $\mathbf{f} \cdot \mathbf{M}$). The resulting value, which reflects the similarity between **f** and **M**, is fed into a two-criterion decision system modeled on the one proposed by Swets and Green (1961). More details can be found in Murdock (1982, 1983), with extensions that allow the system to make latency predictions in Hockley and Murdock (1987). To derive predictions for the decision stage of the model, one needs to know the expectation and variance of the dot-product similarity between probe and memory vector for both old (i.e., previously stored) and new item probes. The derivations reported in Part II are building blocks in the determination of those values.

The same mechanism applies to pair recognition or triple recognition, i.e., situations where the memory system must decide whether or not it has seen a particular pair or triplet before. For pair recognition, the value of the dot product of the convolved pair probe $(\mathbf{f} * \mathbf{g})$ and the memory vector **M** containing stored pairwise associations (i.e., $(\mathbf{f} * \mathbf{g}) \cdot \mathbf{M}$) is fed into the decision system. For triplet recognition, it is the value of dot product $(\mathbf{f} * \mathbf{g} * \mathbf{h}) \cdot \mathbf{M}$, where **M** contains stored triple associations.

The generative processes of recall and redintegration are modeled by the correlation operation. If one thinks of convolution as dispersing or distributing the associative information, then correlation is the recombination of the information to its original form. For associative recall, the probe **f** is correlated with the memory vector **M** (i.e., $\mathbf{f} \# \mathbf{M}$) which contains, among other components, the convolution $(\mathbf{f} * \mathbf{g})$. As outlined in Murdock (1982), all contributions to the memory vector are mutually independent, which means that the correlation of **f** with **M** can be treated as the sum of the correlations of **f** with the components of **M**. The correlation component $\mathbf{f} \# (\mathbf{f} * \mathbf{g})$ results in a vector $\mathbf{g}'$, i.e., a vector whose N central elements are similar to those of vector **g**. The degree of similarity between $\mathbf{g}'$ and **g** is measured by the dot product, i.e., $\mathbf{g} \cdot \mathbf{g}' = \mathbf{g} \cdot (\mathbf{f} \# (\mathbf{f} * \mathbf{g}))$.

Redintegration uses the same operations as recall, with auto-associations as the crucial associative information in the memory vector **M** (i.e., the correlation $\mathbf{f} \# (\mathbf{f} * \mathbf{f})$ has output $\mathbf{f}'$ which is similar to **f** to the extent $\mathbf{f} \cdot (\mathbf{f} \# (\mathbf{f} * \mathbf{f}))$.

To predict recall performance, TODAM computes the probability that the vector retrieved by the correlation operation (e.g., $\mathbf{g}'$) is more similar to the target item **g**

than to any other item and that the retrieved information $\mathbf{g}'$ is within criterial range of the target. (For the exact mathematical expression and more details see Murdock, 1982). Needed for this computation are the expectations and variances of "old item" (i.e., $\mathbf{g} \cdot \mathbf{g}'$) and "new item" (i.e., $\mathbf{h} \cdot \mathbf{g}'$) similarity distributions. The derivations reported in Part II are building blocks in the determination of those values.

PART II: SUMMARY OF EXPECTATION AND VARIANCE DERIVATIONS

In developing expressions for predicted memory performance, TODAM relies on the basic methods of probability theory. Since the elements or micro features of an item or event vector are conceptualized as independent random variables, expectations and variances of the distributions of dot products between item vectors and/or between convolution–correlations of item vectors can be expressed as the sums of products of these independent random variables. The variance expressions in Appendix A are shown in this form. The variance of the dot product of two vectors $\mathbf{f}$ and $\mathbf{g}$, for example, as shown in Appendix A (Ia) consists of the sum of N components each of which is the product of two independent random variables, $\mathbf{X}$ and $\mathbf{Y}$. For most of the variance expressions in Appendix A, not all variance components are independent and covariance terms must be considered. The derivations thus consist of determining the types of variance components occurring in a particular expression, the number of components of each type as a function of N, the dimensionality of the item vectors, and finally the number and types of covariance components.

Appendix B lists the values of such variance and covariance components, i.e., variances and covariances of products of random variables, assuming independent and identically normal distributions of $\mathbf{T}$, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ with mean zero and variance σ^2. These expressions are easily computed using the expectations of powers of $\mathbf{X}$ also shown in Appendix B which are obtained by use of the moment-generating function (e.g., Hoel, 1962). The expectations of odd powers of $\mathbf{X}$ are, of course, zero. This latter fact makes it easy to determine the expectations of the dot-product expressions of Appendix A. Dot-product expressions that are a perfect match, i.e., that have identical components on both sides of the dot-product operation (e.g., $\mathbf{f} \cdot \mathbf{f}$, or $(\mathbf{f} * \mathbf{g} * \mathbf{h}) \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})$), have expected values of one. The one exception to this rule is $(\mathbf{f} * \mathbf{f}) \cdot (\mathbf{f} * \mathbf{f})$, which has an expectation of $(2N + 1)/N$. All other dot-product expressions have expected values (but not variances) that are equal to zero. For cases of partial probe information (e.g., $p\mathbf{f} \cdot \mathbf{f}$), the expected value is p times the expected value of the corresponding complete probe case. The expected value of $(p\mathbf{f} * p\mathbf{f}) \cdot (\mathbf{f} * \mathbf{f})$ in Table 3c is also $p(2N + 1)/N$, where the p attributes present are the same for both $p\mathbf{f}$ vectors. The variances of these dot-product similarity distributions are computed by substituting the values of the variance and covariance components shown in Appendix B into the expressions of Appendix A. After making the additional assumption that $\sigma^2 = 1/N$, discussed in Part I, the expressions can be rearranged and expressed as a function of N. In this form they

are shown in Tables 1 to 6. Some of the variances in part (a) of Tables 1, 2, and 3 already appear in Murdock (1982) but are repeated here for completeness.

The explicit component expressions as shown in Appendix A are useful because of their generality; i.e., they will facilitate the development of expressions for the moments of item resemblance distributions when making assumptions different from the ones employed by TODAM (e.g., different distributional assumptions about the component features, or the use of correlated rather than independent random vectors to model item similarity). A different set of assumptions will result in different values for the component expectations, variances, and covariances listed in Appendix B. Once these component expressions have been worked out (a relatively simple task using, for example, such references as Goodman (1960, 1962) and Bohrnstedt and Goldberger (1969) for the generalization of TODAM to the correlated vector case), they can be directly substituted into the formulae provided in Appendix A, thus simplifying the derivation of future item resemblance distribution moments.

Analytic expressions for the moments of the model's resemblance distributions are clearly more time and cost efficient than the use of computer simulation. Simulations for recall or recognition involving item triplets can take many hours of mainframe CPU time even for small values of N. (Computation time can be reduced by moving into the Fourier transform domain where the convolution operation maps into the multiplication operation, but is still significant.) However, computer simulation is valuable in validating the analytic derivations where it is all too easy to leave out the occasional variance or covariance component. For this purpose, Monte Carlo simulations were conducted for all dot-product resemblance expressions of Appendix A, factorially varying parameter N ($N = 15, 95, 195$) and parameter p ($p = 0.2, 0.4, 0.6, 0.8, 1.0$). A total of 500 replications of a particular computation went into the construction of the distribution of that resemblance expression over trials, at which point its expectation and variance were computed. Moments obtained by simulation were compared with those predicted by the analytic expressions. Predicted and obtained values for all expressions agreed within conventional standard error bounds for all combinations of p and N.

The reader will note the absence of "recall" expressions of the type $\mathbf{g} \cdot (\mathbf{f} \# (\mathbf{f} * \mathbf{g}))$ in Tables 1 to 6. This is due to the "recall-recognition identity" of the model (Murdock, 1982, 1985). By the way convolution and correlation are defined, the "recall" expression $\mathbf{g} \cdot (\mathbf{f} \# (\mathbf{f} * \mathbf{g}))$, for example, can be rearranged to become "recognition" expression $(\mathbf{f} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{g})$,

$$\begin{aligned}
\mathbf{g} \cdot (\mathbf{f} \# (\mathbf{f} * \mathbf{g})_x)_x &= \sum_i g(x) \sum_i \sum_{i'} f(i) f(x+i-i') g(i') \\
&= \sum_i \left[\sum_i g(z-i) f(i) \right] \left[\sum_{i'} f(z-i') g(i') \right], \qquad \text{where } z = x+i \\
&= (\mathbf{f} * \mathbf{g})_z \cdot (\mathbf{f} * \mathbf{g})_z. \qquad (5)
\end{aligned}$$

Thus the two resemblance distributions and their moments are identical.

Another parameter of TODAM, not previously mentioned, is p, the proportion of item vector features. Parameter p serves a double role. First, it represents the proportion of item features present in a degraded presentation of a particular item when modeling redintegration by the memory system. The resemblance distributions involved in modeling this process are of the type $\mathbf{f} \cdot (p\mathbf{f} \# (\mathbf{f} * \mathbf{f})) = (p\mathbf{f} * \mathbf{f}) \cdot (\mathbf{f} * \mathbf{f})$. Its second function is connected to an additional encoding assumption that enables TODAM to model learning, that is, the improvement of memory performance with repeated presentation of the same item. The additional assumption is probabilistic encoding, that is, the idea that only a certain, random proportion of item features, $p\mathbf{f}$, is encoded at any presentation of the item. Repeated presentation will improve performance because increasingly more item features become encoded (Murdock & Lamon, 1988). Resemblance distributions involved in modeling probabilistic encoding are of the type $\mathbf{h} \cdot [(\mathbf{f} * \mathbf{g}) \# (p\mathbf{f} * \mathbf{g} * \mathbf{h})] = (p\mathbf{f} * \mathbf{g} * \mathbf{h}) \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})$.

The remainder of Part II reviews the modeling applications of the variance expressions in Tables 1 to 6.

Table 1 lists the variances of resemblance distributions necessary to model item recognition. The expressions $\mathbf{f} \cdot \mathbf{f}$ and $\mathbf{f} \cdot \mathbf{g}$ are the components of the recognition operation $\mathbf{f} \cdot \mathbf{M}$ that "compare" the probe with the item information stored in $\mathbf{M}$. Stored item information can either match the probe ($\mathbf{f} \cdot \mathbf{f}$) or mismatch ($\mathbf{f} \cdot \mathbf{g}$). Part (b) of Table 1 generalizes the derivations to either the presentation of a degraded probe ($p\mathbf{g}$) or the probabilistic encoding of the item into the memory vector ($p\mathbf{f}$), since $p\mathbf{g} \cdot \mathbf{f} = \mathbf{g} \cdot p\mathbf{f}$.

Table 2 applies to the recognition of item information encoded in a paired associate learning context. That is, it lists the variances of resemblance distributions contained in $\mathbf{f} \cdot \mathbf{M}$, where $\mathbf{M}$ contains associative information. If encoding consists of $\mathbf{M}_j = \alpha \mathbf{M}_{j-1} + \gamma_1 \mathbf{f}_j + \gamma_2 \mathbf{g}_j + \omega(\mathbf{f}_j * \mathbf{g}_j)$, one needs to know the resemblance distribution variances in Table 1 as well as $\mathbf{f} \cdot (\mathbf{f} * \mathbf{g})$, $\mathbf{h} \cdot (\mathbf{f} * \mathbf{g})$, supplied in Table 2, to

TABLE 1

Variances of Single-Item Probe and Single-Item Trace Matches for (a) Integral Probe and (b) Degraded Probe

Probe	Trace $\mathbf{f}$
(a)	
$\mathbf{f}$	$2/N$
$\mathbf{g}$	$1/N$
(b)	
$p\mathbf{f}$	$2p/N$
$p\mathbf{g}$	p/N

TABLE 2

Resemblance Distribution Variances for Single-Item Probe and Convolved-Pair Trace Matches for (a) Integral Probe and (b) Degraded Probe

Probe	Trace: $\mathbf{f} * \mathbf{f}$	Trace: $\mathbf{f} * \mathbf{g}$
(a)		
$\mathbf{f}$	$\frac{11N^2+17N-1}{2N^3}$	$\frac{7N^2+4N+1}{4N^3}$
$\mathbf{h}$	$\frac{7N^2-5}{4N^3}$	$\frac{3N^2+1}{4N^3}$
(b)		
$p\mathbf{f}$	$\frac{8p^2N^2+3pN^2+17pN-8p^2+7p}{2N^3}$	$\frac{4p^2N^2+3pN^2+4pN+p}{4N^3}$
$p\mathbf{h}$	$\frac{p(7N^2-5)}{4N^3}$	$\frac{p(3N^2+1)}{4N^3}$

determine the variance of $\mathbf{f} \cdot \mathbf{M}$. Similarly, the variances of the resemblance distributions $\mathbf{f} \cdot (\mathbf{f} * \mathbf{f})$ and $\mathbf{h} \cdot (\mathbf{f} * \mathbf{f})$ are required for cases where the associative information consists of auto-associations. As in Table 1, part (b) of Table 2 generalizes the derivations to situations of degraded probe information or probabilistic encoding.

There are four areas of application for Table 3. The first is recognition of an item pair after a paired associate learning task. To determine the variance of $(\mathbf{f} * \mathbf{g}) \cdot \mathbf{M}$, where $\mathbf{M}$ is as described for Table 2, one needs to know the variances of such resemblance distributions as $(\mathbf{f} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{g})$, $(\mathbf{f} * \mathbf{h}) \cdot (\mathbf{f} * \mathbf{g})$, or $(\mathbf{h} * \mathbf{j}) \cdot (\mathbf{f} * \mathbf{g})$. The second application is cued recall of one member of a pair with the help of its associate. As shown in Part II, $\mathbf{g} \cdot (\mathbf{f} \# \mathbf{M}) = (\mathbf{f} * \mathbf{g}) \cdot \mathbf{M}$, with the same variance components as those in the first application. The third area of application is redintegration of item information where $\mathbf{M}$ contains auto-associative information such as $(\mathbf{f} * \mathbf{f})$ or $(\mathbf{g} * \mathbf{g})$. To determine the variance of expressions such as $\mathbf{f} \cdot (\mathbf{f} \# \mathbf{M})$, one needs to know the variances of resemblance distributions such as $(\mathbf{f} * \mathbf{f}) \cdot (\mathbf{f} * \mathbf{f})$, $(\mathbf{f} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{f})$, and $(\mathbf{g} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{f})$. The final application is the recovery of serial-order information from a memory vector $\mathbf{M}$ that contains associative information of the type $(\mathbf{f}_j * \mathbf{f}_{j-1})$, where j denotes the serial position of an item. To determine the variance of expressions such as $\mathbf{f}_j \cdot (\mathbf{f}_{j-1} \# \mathbf{M})$, one needs to know the variances of resemblance distributions shown in Table 3.

Again, part (b) of Table 3 generalizes part (a) to situations of degraded probe information or probabilistic encoding. These variance components are needed to model such tasks as word-fragment completion, cued recall with partial cues, or the tip of the tongue phenomenon (i.e., cases where a degraded probe must be redin-

TABLE 3

Resemblance Distribution Variances for Convolved-Pair Probe and Convolved-Pair Trace Matches for (a) Integral Probe, (b) Partially Degraded Probe, and (c) Degraded Trace

	Trace	
Probe	**f** * **f**	**f** * **g**
(a)		
f * **f**	$\frac{112N^3+156N^2+8N+12}{3N^4}$	
f * **g**		$\frac{16N^3+6N^2+2N}{3N^4}$
f * **h**	$\frac{32N^3-21N^2+76N+3}{6N^4}$	$\frac{5N^3+3N^2+N}{3N^4}$
g * **g**	$\frac{8N^3+6N^2+4N+6}{3N^4}$	
h * **j**	$\frac{8N^3+3N^2+4N+3}{6N^4}$	$\frac{2N^3+N}{3N^4}$
(b)		
p**f** * **f**	$\frac{1}{12N^4}(288p^3N^3+48p^3N^3+112pN^3+36p^3N^2-57p^2N^2+609pN^2+36N^2+24p^2N+89pN-81N-72p^2+75p+45)$	$\frac{1}{6N^4}(22p^2N^3+4pN^3+6N^3-15p^2N^2+12pN^2-18N^2-22p^2N+86pN+12N-3p^2+6p)$
p**f** * **g**	$\frac{24p^2N^3+8pN^3-21pN^2+28pN+48N+3p}{6N^4}$	$\frac{1}{3N^4}(8p^2N^3+8pN^3-3p^2N^2+9pN^2+p^2N+pN)$
f * p**g**	$\frac{p(32N^3-21N^2+76N+3}{6N^4}$	
p**f** * **h**		$\frac{3p^2N^3+2pN^3+3pN^2+pN}{3N^4}$
f * p**h**		$\frac{p(5N^3+3N^2+N)}{3N^4}$
p**g** * **g**	$\frac{1}{12N^4}(24p^2N^3+8pN^3-9p^2N^2+33pN^2-6p^2N+22pN-9p^2+33p)$	
p**h** * **j**	$\frac{p(8N^3+3N^2+4N+3}{6N^4}$	$\frac{p(2N^3+N)}{3N^4}$

Table continued

TABLE 3 (*continued*)

	Trace $p\mathbf{f} * p\mathbf{f}$
(c)	
$\mathbf{f} * \mathbf{f}$	$\frac{1}{12N^4}(96p^4N^3 + 224p^3N^3 + 32p^2N^3 + 96pN^3 - 432p^4N^2 + 315p^3N^2 + 114p^2N^2 + 591pN^2 + 36N^2 + 480p^4N - 320p^3N - 56p^2N - 72N - 144p^4 + 96p^3 - 48p^2 + 108p + 36)$
$\mathbf{f} * \mathbf{g}$	$\frac{1}{6N^4}(32p^2N^3 - 6p^2N^2 - 15pN^2 - 44p^2N + 120pN - 6p^2 + 9p)$
$\mathbf{g} * \mathbf{g}$	$\frac{1}{12N^4}(32p^2N^3 - 12p^2N^2 + 36pN^2 - 8p^2N + 24pN - 12p^2 + 36p)$
$\mathbf{g} * \mathbf{h}$	$\frac{1}{6N^4}(8p^2N^3 - 6p^2N^2 + 9pN^2 + 4p^2N - 6p^2 + 9p)$

TABLE 4

Resemblance Distribution Variances for Single-Item Probe and Convolved-Triplet Trace Matches for (a) Integral Probes and (b) Degraded Probes

Probe	Trace $\mathbf{f} * \mathbf{g} * \mathbf{h}$
(a)	
$\mathbf{f}$	$\frac{5N^3 + 3N^2 + N}{3N^4}$
$\mathbf{j}$	$\frac{2N^3 + N}{3N^4}$
(b)	
$p\mathbf{f}$	$\frac{3p^2N^3 + 2pN^3 + 3pN^2 + pN}{3N^4}$
$p\mathbf{j}$	$\frac{p(2N^3 + N)}{3N^4}$

tegrated), or performance after a sub-optimal study phase (e.g., study at a rapid presentation rate), i.e., cases that result in incomplete encoding of information.

Table 4 is needed to model the recognition of item information that has been encoded in the context of a learning task requiring the association of triplets of information. That is, it lists the variances of resemblance distributions contained in $\mathbf{f} \cdot \mathbf{M}$, where $\mathbf{M}$ contains associative information of the type $(\mathbf{f} * \mathbf{g} * \mathbf{h})$. (Items **f**, **g**, and **h** could represent the subject, predicate, and object of a proposition, or simply three words presented as a triplet on a study list.) To model item recognition performance, one needs to know the resemblance distribution variances in Table 4 (as well as those in Tables 1 and 2 depending on one's encoding assumptions, i.e., if one

TABLE 5

Resemblance Distribution Variances for Convolved-Pair Probe and Convolved-Triplet Trace Matches for (a) Integral Probes and (b) Partially Degraded Probes

Probe	Trace $\mathbf{f} * \mathbf{g} * \mathbf{h}$
(a)	
$\mathbf{f} * \mathbf{g}$	$\frac{595N^4 + 384N^3 + 722N^2 - 192N + 27}{192N^5}$
$\mathbf{f} * \mathbf{f}$	$\frac{230N^4 + 576N^3 + 820N^2 + 192N + 294}{192N^5}$
$\mathbf{f} * \mathbf{j}$	$\frac{259N^4 + 98N^2 + 27}{192N^5}$
$\mathbf{j} * \mathbf{k}$	$\frac{115N^4 + 50N^2 + 27}{192N^5}$
(b)	
$p\mathbf{f} * \mathbf{g}$	$\frac{1}{192N^5}(336p^2N^4 + 259pN^4 + 192p^2N^3 + 192pN^3 + 48p^2N^2 + 674pN^2 - 192pN + 27p)$
$p\mathbf{f} * \mathbf{f}$	$\frac{p(230N^4 + 576N^3 + 820N^2 + 192N + 294}{192N^5}$
$p\mathbf{f} * \mathbf{j}$	$\frac{1}{192N^5}(144p^2N^4 + 115pN^4 + 48p^2N^2 + 50pN^2 + 27p)$
$\mathbf{f} * p\mathbf{j}$	$\frac{p(259N^4 + 98N^2 + 27)}{192N^5}$
$p\mathbf{j} * \mathbf{k}$	$\frac{p(115N^4 + 50N^2 + 27)}{192N^5}$

TABLE 6

Resemblance Distribution Variance for
Convolved-Triplet Probe and Convolved-Triplet Trace Matches for
(a) Integral Probes and (b) Partially Degraded Probes

Probe	Trace $\mathbf{f} * \mathbf{g} * \mathbf{h}$
(a)	
$\mathbf{f} * \mathbf{g} * \mathbf{h}$	$\dfrac{486N^5 + 960N^4 - 30N^3 + 120N^2 + 24N}{60N^6}$
$\mathbf{f} * \mathbf{g} * \mathbf{j}$	$\dfrac{143N^5 + 260N^4 + 85N^3 + 40N^2 + 12N}{60N^6}$
$\mathbf{f} * \mathbf{j} * \mathbf{k}$	$\dfrac{73N^5 + 40N^4 + 35N^3 + 20N^2 + 12N}{60N^6}$
$\mathbf{j} * \mathbf{k} * \mathbf{l}$	$\dfrac{33N^5 + 15N^3 + 12N}{60N^6}$
(b)	
$p\mathbf{f} * \mathbf{g} * \mathbf{h}$	$\dfrac{1}{60N^6}(313p^2N^5 + 173pN^5 + 340p^2N^4 + 560pN^4 + 60N^4 - 295p^2N^3 + 295pN^3 - 30N^3 + 110p^2N^2 + 70pN^2 - 60N^2 + 12p^2N - 18pN + 30N)$
$p\mathbf{f} * \mathbf{g} * \mathbf{j}$	$\dfrac{1}{60N^6}(70p^2N^5 + 73pN^5 + 90p^2N^4 + 170pN^4 + 20p^2N^3 + 65pN^3 + 40pN^2 + 12pN)$
$\mathbf{f} * \mathbf{g} * p\mathbf{j}$	$\dfrac{p(143N^5 + 260N^4 + 85N^3 + 40N^2 + 12N}{60N^6}$
$p\mathbf{f} * \mathbf{j} * \mathbf{k}$	$\dfrac{1}{60N^6}(40p^2N^5 + 33pN^5 + 40pN^4 + 20p^2N^3 + 15pN^3 + 20pN^2 + 12pN)$
$\mathbf{f} * \mathbf{j} * p\mathbf{k}$	$\dfrac{p(73N^5 + 40N^4 + 35N^3 + 20N^2 + 12N}{60N^6}$
$p\mathbf{j} * \mathbf{k} * \mathbf{l}$	$\dfrac{p(33N^5 + 15N^3 + 12N)}{60N^6}$

assumes that item and pairwise-associative information is stored in the same vector). Expression $\mathbf{f} \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})$ provides the item—triplet resemblance component of $\mathbf{f} \cdot \mathbf{M}$ that constitutes a "match"; $\mathbf{j} \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})$ stands for all those components of $\mathbf{f} \cdot \mathbf{M}$ that constitute "mismatches."

Table 5 does for pair recognition what Table 4 does for item recognition. It provides the variances of resemblance distributions needed to model recognition of pairs of items that were encoded as a component of a triplet of information (i.e., recognize $(\mathbf{f} * \mathbf{g})$ when $\mathbf{M}$ contains such expressions as $(\mathbf{f} * \mathbf{g} * \mathbf{h})$). In particular, one needs the resemblance distribution variance for a "match," $(\mathbf{f} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})$; a partial "match," $(\mathbf{f} * \mathbf{j}) \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})$; and a "mismatch," $(\mathbf{j} * \mathbf{k}) \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})$.

There are three areas of application for Table 6. The first is the recognition of triple associates that were encoded as such, i.e., $(\mathbf{f} * \mathbf{g} * \mathbf{h}) \cdot \mathbf{M}$, where $\mathbf{M}$ contains expressions such as $(\mathbf{f} * \mathbf{g} * \mathbf{h})$. Again, variances of complete "match," partial "match" (containing one or two matching elements), and "mismatch" resemblance distributions are needed. The second and third applications are for modeling cued recall after a triplet learning task, with a single item or a pair probe, respectively. If $\mathbf{M}$ contains the triple association $(\mathbf{f} * \mathbf{g} * \mathbf{h})$, then retrieval with item probe $\mathbf{f}$, i.e., $\mathbf{f} \# \mathbf{M}$, produces output $(\mathbf{g} * \mathbf{h})'$, a vector resembling the convolution $(\mathbf{g} * \mathbf{h})$, where resemblance can be measured by the dot product of the two vectors, $(\mathbf{g} * \mathbf{h}) \cdot (\mathbf{g} * \mathbf{h})' = (\mathbf{g} * \mathbf{h}) \cdot (\mathbf{f} \# \mathbf{M}) = (\mathbf{f} * \mathbf{g} * \mathbf{h}) \cdot \mathbf{M}$. To determine the variance of this expression, the variances of the resemblance distributions listed in Table 6 are required. The same is true for recall of a single element of a triplet using the other two elements as a recall cue. The retrieval operation $(\mathbf{f} * \mathbf{g}) \# \mathbf{M}$ produces output $\mathbf{h}'$, a vector resembling $\mathbf{h}$, where resemblance is measured by $\mathbf{h} \cdot \mathbf{h}' = \mathbf{h} \cdot (\mathbf{f} * \mathbf{g}) \# \mathbf{M} = (\mathbf{f} * \mathbf{g} * \mathbf{h}) \cdot \mathbf{M}$. Again, part (b) of Table 6 generalizes the derivations for cases of incomplete probe or trace information.

PART III: THREE SAMPLE APPLICATIONS

This section provides a step-by-step tutorial on how to apply the expressions derived in the previous section to model sets of data. The tutorial is centered around three examples, selected to illustrate the full range of applications that can be modeled using the expressions of item resemblance distribution moments provided. The first two data sets require the assumption that only a subset, $p\mathbf{f}$, of item micro features gets represented during encoding (Loftus, 1974) or retrieval (Tulving & Watkins, 1973). The third example (Clark & Shiffrin, 1987) illustrates the use of triple association expressions, as well as the modeling of different decision rule instructions. Data sets that have these characteristics are relatively scarce. The selection of examples was guided, in addition, by the requirement that they not be so complex that the modeling of particular experimental details would obscure demonstration of the basic procedure. Thus, the first data set may appear to be a "toy problem," but the simplicity and brevity of experimental procedure and data

allow one to demonstrate basic model application of TODAM without having to deal with too many idiosyncratic experimental details. The full power of a multi-parameter model like TODAM can, of course, only be appreciated when larger and more complex sets of data are fit with the same small number of parameters. Such model fits are reported elsewhere (see Hockley & Murdock, 1987; Murdock, 1987a, b; Murdock & Lamon, 1987). The goal of the work reported in this paper is to facilitate the basic model application procedures (by providing analytic expression of the item resemblance distribution moments and by demonstrating their application with simple examples) so that researchers can concentrate instead on the particular encoding and retrieval representation assumptions when they fit a model like TODAM to a complex data set. It is the felicity of these "higher-order" assumptions about encoding and retrieval details that determines to a considerable degree the degree of fit of a model like TODAM.

Cued Recall with Partial Item Probes

Tulving and Watkins (1973), in an attempt to advance the continuity view of recall and recognition as essentially the same operation, report a simple experiment where subjects viewed a 28-item list of five-letter words presented on a TV screen one word at a time at a rate of 2 sec per word. Subjects were instructed to memorize the words for the subsequent test phase where they would receive a list of clues (one for each word) designed to help them recall the words on the study list. Instructions emphasized that only words that were remembered as having been on the list were to be written down alongside the cue and that no guesses were allowed. Cues consisted of either the first two, three, four, or all five letters of a particular target word, with cue condition by word combinations counterbalanced across subjects. (There are more details to the study, including a free recall condition for the purpose of testing the authors' continuity hypothesis, which are of no concern here. The sole reason for reporting the study is to illustrate how the derivations in resemblance distribution moments reported in Part II can be used to model experimental results, in this case cued recall with partial cues, with a convolution–correlation distributed memory model.)

The first modeling step involves the decision of what information gets encoded during the experiment into the memory vector **M**. Given single-item presentation and the instructions that subjects received, it seems reasonable to assume that only item or auto-associative information about the study words was encoded. To model partially cued recall, or redintegration, we must adopt a version of the correlation–convolution model that contains auto-associative information such as $\mathbf{f} * \mathbf{f}$ or $\mathbf{g} * \mathbf{g}$. Since no cross-item associations are required, the simplest assumption is that the memory vector **M** contains *only* auto-associative information, i.e.,

$$\mathbf{M}_j = \alpha \mathbf{M}_{j-1} + (\mathbf{f}_j * \mathbf{f}_j). \tag{6}$$

The proportions of words subjects recalled as a function of the length of the cue fragment are shown in Table 7 (as read off Fig. 1 in Tulving and Watkins (1973)). The second modeling step involves the decision of how to represent retrieval cues. We will model partial cue information by $p\mathbf{f}$, where $p < 1$. There is no reason to believe that p has anything but a monotonic relationship to the proportion of letters presented (i.e., the first two, three, four, or five). Thus, the model parameters that must be estimated for this application are p, the proportion of cue features; α, the forgetting parameter; N, the number of item features; and a and b, the upper and lower tolerance limits of TODAM's two-criterion decision system.

The overall logic of the model fit was to determine the optimal values of α, N, a, and b to model performance for the "intact cue" condition (i.e., number of cue letters is 5). These parameters were then fixed to these values for the second part of the fit that determined what values of p would produce the reported performance levels for the three partial cue conditions. As can be seen in Table 7, TODAM provides a perfect fit of the data. Parameter values for this fit were $\alpha = 0.98$, $N = 100$, $a = 1.0$, $b = 3.0$, with the values of p for the various conditions listed in Table 7. According to TODAM, a cue consisting of the first two of five letters contains about 50% of the stimulus information that would be encoded for the intact cue. This percentage increases to 64 and 74% respectively, for three- and four-letter fragments.

To obtain a perfect fit for four data points with a seven-parameter model is not surprising. It should be noted however, that TODAM, in fact, provides predictions for the percentage of correct recall for every one of the 28 serial positions at each of the four cue conditions. Thus, seven parameters actually predict 112 data points. (Because Tulving and Watkins do not break their results down by serial position, TODAM's predictions were averaged across serial positions to model the data reported.) It should also be noted that the parameter estimates obtained closely resemble those obtained in independent fits of other data sets.

To compute this model fit, one must know the values of the expectation and

TABLE 7

Observed Proportions of Words Recalled for Cued Recall Task (Tulving & Watkins, 1973), and Performance Predicted by Convolution–Correlation Model

Number of cue letters	Observed proportion of words recalled	Predicted proportion of words recalled[a]	Estimate of p[b]
2	0.28	0.283	0.50
3	0.56	0.567	0.64
4	0.70	0.704	0.74
5	0.85	0.854	1.00

[a] Values of model parameters for this fit are $\alpha = 0.98$, $N = 100$, $a = 1.0$, and $b = 3.0$.
[b] The proportions of cue features, p_i, are estimated as i free parameters.

variance of both "signal" and "noise" distributions. The "signal" distribution, in this case, refers to the resemblance of the information retrieved with the help of the cue to the target item, i.e., $\mathbf{f} \cdot (p\mathbf{f} \# \mathbf{M})$, where $\mathbf{M}$ is as defined in Eq. (6). Thus,

$$\mathbf{f} \cdot (p\mathbf{f} \# \mathbf{M}) = (\mathbf{f} * p\mathbf{f}) \cdot \mathbf{M} = (\mathbf{f} * p\mathbf{f}) \cdot [c_j(\mathbf{f} * \mathbf{f}) + k(\mathbf{g} * \mathbf{g})],$$

where

$$k = \sum_{j=1}^{m} \alpha^{m-j} - c_j. \tag{7}$$

In this equation, $c_j = \alpha^{(m-j)}$, with m denoting the list length and j the serial position the cue item had on the list. Using the expressions for the expectation and variance of the component resemblances $(\mathbf{f} * p\mathbf{f}) \cdot (\mathbf{f} * \mathbf{f})$ and $(\mathbf{f} * p\mathbf{f}) \cdot (\mathbf{g} * \mathbf{g})$ derived in Part II, and weighting them by the appropriate constants, we can express the expectation and variance of the "signal" distribution as follows:

$$\begin{aligned} E[\mathbf{f} \cdot (p\mathbf{f} \# \mathbf{M})] &= c_j p(2N+1)/N \\ \mathrm{Var}[\mathbf{f} \cdot (p\mathbf{f} \# \mathbf{M})] &= c_j(288p^3N^3 + 48p^2N^3 + 112pN^3 + 36p^3N^2 \\ &\quad - 57p^2N^2 + 609pN^2 + 36N^2 + 24p^2N + 89pN \\ &\quad - 81N - 72p^2 + 75p + 45)/12N^4 \\ &\quad + kp(8N^3 + 6N^2 + 4N + 6)/3N^4. \end{aligned} \tag{8}$$

The "noise" distribution for this example refers to the resemblance of information retrieved with the help of the cue to an intra-list intrusion item, i.e., $\mathbf{g} \cdot (p\mathbf{f} \# \mathbf{M})$. Since

$$\mathbf{g} \cdot (p\mathbf{f} \# \mathbf{M}) = (\mathbf{g} * p\mathbf{f}) \cdot \mathbf{M} = (\mathbf{g} * p\mathbf{f}) \cdot [c_j(\mathbf{f} * \mathbf{f}) + k(\mathbf{g} * \mathbf{g})],$$

where again

$$k = \sum_{j=1}^{m} \alpha^{m-j} - c_j, \tag{9}$$

the expectation and variance of the "noise" distribution can be expressed in terms of the model parameters with the help of the moments of component resemblances $(\mathbf{g} * p\mathbf{f}) \cdot (\mathbf{f} * \mathbf{f})$ and $(\mathbf{g} * p\mathbf{f}) \cdot (\mathbf{g} * \mathbf{g})$ derived in Part II, again weighted by the appropriate constants:

$$\begin{aligned} E[\mathbf{g} \cdot (p\mathbf{f} \# \mathbf{M})] &= 0 \\ \mathrm{Var}[\mathbf{g} \cdot (p\mathbf{f} \# \mathbf{M})] &= c_j(24p^2N^3 + 8pN^3 - 21pN^2 + 28pN)/6N^4 \\ &\quad + kp(32N^3 - 21N^2 + 76N + 3)/6N^4. \end{aligned} \tag{10}$$

To predict recall performance with TODAM, one needs to know the probability that the retrieved information $\mathbf{f}'$ is more similar to the target $\mathbf{f}$ than to any other item and that it is within criterial range of the target. Thus, the probability of correct recall can be computed as

$$P_{\text{correct}} = \int_a^b \phi_o(s) \prod_j \left[1 - \int_{1-s}^{1+s} \phi_{n,j}(s')\, ds' \right] ds, \tag{11}$$

where the product $\prod$ runs from $j = 1$ to $(m-1)$, $\phi_o(s)$ denotes the old or "signal" similarity distribution, and $\phi_{n,j}$ denotes the $(m-1)$ new or "noise" similarity distributions. These distributions are indexed by s because their moments are a function of serial position. To compute the probability of correct recall averaged over serial position as reported by Tulving and Watkins (1973), one simply averages the outputs of Eq. (11) for each of the m serial positions.

RECOGNITION OF RAPIDLY PRESENTED ITEMS

Loftus (1974) reports a study designed to test the hypothesis that "verbal" stimuli (i.e., nouns) undergo qualitatively different encoding processes than "nonverbal" stimuli (i.e., geometric line patterns). Again, the study's hypothesis or any conclusions are of minor concern here. We will use some of the study's data simply to illustrate how this type of data could be modeled by a convolution–correlation memory model.

The study consisted of the rapid sequential presentation of a 16-item list on a CRT screen, where the items were either common nouns (Experiment 1) or random geometric line patterns (Experiment 2). Exposure times per item varied randomly within the study list, with the restriction that two of each of the possible times shown in Table 8 occurred per list. Subjects were told to expect a yes–no recognition test task following the study phase. In the test phase, the 16 target stimuli were presented together with an equal number of new foils drawn from the same population of stimuli, one at a time, for an exposure duration of 2 sec per item. The values of the signal detection memory strength measure d' for the various exposure durations of Experiments 1 and 2 (read off Figs. 1 and 3 in Loftus (1974)) can also be found in Table 8. (The study also analyzed d' as a function of the serial position of the test items on the study list, but this aspect of the data will not concern us here. TODAM has no trouble fitting serial position curves (see Murdock, 1985), and this discussion need not be repeated here.)

The critical assumption in modeling these recognition data is that limited exposure duration results in incomplete encoding of the study items, i.e., only a fraction, p, of item features will be encoded. This fraction p will be some function of the exposure duration t_p, with the minimum assumption of a monotonic relationship. (Some of Loftus' hypotheses about qualitative encoding differences

TABLE 8

Observed d' as a Function of Exposure Time in an Item Recognition Task for (a) Word Stimuli and (b) Geometric Pattern Stimuli (Loftus, 1974) and Performance Predicted by Convolution–Correlation Model

Exposure duration t_p (msec)	Observed d'	Predicted d'[a]	Estimate of p[b]
(a)			
50	0.05	0.03	0.0009
75	0.20	0.23	0.014
100	0.40	0.43	0.022
150	0.78	0.70	0.079
200	0.90	0.89	0.120
350	1.15	1.22	0.232
500	1.42	1.40	0.330
(b)			
125	0.16	0.01	0.0003
250	0.19	0.21	0.010
500	0.57	0.52	0.023
1000	0.80	0.83	0.200
1500	0.97	0.99	0.301
2000	1.11	1.09	0.389

[a] Values of model parameters for this fit are $\alpha = 0.53$ and $N = 222$ (part a) and $\alpha = 0.40$ and $N = 234$ (part b).

[b] Estimated by a two-parameter function of exposure duration t_p, i.e., $p = 1 - \exp(-b(t_p - t_0))$, with $b = 0.906$ and $t_0 = 59$ msec (part a) and $b = 0.270$ and $t_0 = 175$ msec (part b).

between "verbal" and "nonverbal" stimuli could be expressed as differences in the function relating p to t_p.) For the fits reported here, the stronger assumption that the relationship between p and t_p can be modeled by a cumulative exponential function, i.e., $p = 1 - e^{-bt}$, was made. This assumption allows one to estimate any number of values of p with just one free parameter, b. Another parameter was added by assuming a positive-valued time intercept, t_0, so that $t = t_p - t_0$.

The expectation and variance of both the "signal" (new item) and "noise" (old item) distributions are required to compute model predictions for the values of d' reported by Loftus. Because of an equal number of new and old item presentations at test, d' is computed as

$$d' = (E(\text{Signal}) - E(\text{Noise}))/\sqrt{(\text{Var}(\text{Signal}) + \text{Var}(\text{Noise}))/2}. \tag{12}$$

The "signal" in this case is the old item dot-product resemblance $\mathbf{f} \cdot (\mathbf{f} \# \mathbf{M}) = (\mathbf{f} * \mathbf{f}) \cdot \mathbf{M} = (\mathbf{f} * \mathbf{f}) \cdot [c_f(p\mathbf{f} * p\mathbf{f}) + k(p\mathbf{g} * p\mathbf{g})]$, where p is a function of t_p and t_0 as described in the last paragraph and c and k are defined as in the last section (see Eq. (7)). The expectation of this old item resemblance distribution thus is equal to

$c_j E[(p\mathbf{f} * p\mathbf{f}) \cdot (\mathbf{f} * \mathbf{f})]$ and its variance is equal to $c_j \operatorname{Var}[(p\mathbf{f} * p\mathbf{f}) \cdot (\mathbf{f} * \mathbf{f})] + k \operatorname{Var}[(p'\mathbf{g} * p'\mathbf{g}) \cdot (\mathbf{f} * \mathbf{f})]$, where p' represents the p corresponding to the average exposure duration, t', of stimuli during study (i.e., $p' = 1 - e^{-b(t'-t_0)}$). The "noise" is the new item dot-product resemblance $\mathbf{k} \cdot (\mathbf{h} \# \mathbf{M}) = (\mathbf{k} * \mathbf{h}) \cdot \mathbf{M} = (\mathbf{k} * \mathbf{h}) \cdot [c_j(p\mathbf{f} * p\mathbf{f}) + k(p\mathbf{g} * p\mathbf{g})]$. The expectation of this distribution is zero and its variance is equal to $c_j \operatorname{Var}[(p\mathbf{f} * p\mathbf{f}) \cdot (\mathbf{k} * \mathbf{h})] + k \operatorname{Var}[(p'\mathbf{g} * p'\mathbf{g}) \cdot (\mathbf{k} * \mathbf{h})]$. Combining the component variances of Table 3c with appropriate weights, one can express the variances of the old and new item distributions as a function of four model parameters, the forgetting parameter, α; the number of item features, N; and b and t_0 of the function relating p to t_p. The values of these parameters for the fits reported in Table 8 are listed at the bottom of that table. It may be noted that the difference in d' between the word stimuli (part (a)) and geometric pattern stimuli (part (b)) is captured by a different function relating p to t_p (i.e., different values of b and t_0), whereas α and N are similar for both conditions. A measure of goodness-of-fit between expected and observed values can be obtained by computing the proportion of variance accounted for as $\{1 - [\sum(o-e)^2/\sum(o-\bar{o})^2]\}$, where $\bar{o}$ stands for the arithmetic mean of the observed values. Using this measure, TODAM accounts for 99.1% of the variance in part (a) and for 96.5% of the variance in part (b).

The data reported and modeled here are d' values averaged over several subjects. The same modeling could, of course, be applied to individual subject data. Individual differences can then be expressed by differences in model parameters.

It should be emphasized that the fits reported are only first approximations of the fits that could be obtained with more sophisticated modeling. For the Loftus data, for example, the fits could probably be substantially improved by incorporating output interference into the model. (Such improvements, however, go beyond the purpose of this paper.) The three sample applications developed here should therefore be regarded only as basic illustrations of how such data could be modeled, and not as the last word about the degree to which TODAM or other convolution–correlation models could fit these data.

Recognition of Single Items, Pairs, or Triplets after Triple Association Encoding

Clark and Shiffrin (1987) report data on the recognition performance (probability of a forced choice "old" classification as the dependent variable) for multiple-item study and test probes. In their experiment, subjects studied 21 consecutive triplets (presented for 5 sec each) and were then presented with the types of probes listed in Table 9. These probes were triplets, pairs, or singletons, the components of which could either all come from the same "old" triplet (e.g., ABC or AB), come from "old" but different triplets (e.g., ABC' or $AB'C''$, with "different triplet" denoted by the prime symbol), or be partially or completely "new" items (e.g., ABX, $AB'X$, or XY). Subjects made "old–new" decisions on each trial following one of three decision rules which was disclosed to the subject after the study

TABLE 9

Observed Proportions of "Old" Judgment for Different Probe Triplets and Three Decision Rules in Old–New Recognition Task (Clark & Shiffrin, 1987) and Performance Predicted by Convolution-Correlation Model

	Decision rule					
	Intact		Any-old		All-old	
Probe triplet	P_{obs}	P_{pred}[a]	P_{obs}	P_{pred}[b]	P_{obs}	P_{pred}[c]
ABC	0.73	0.721	0.88	0.852	0.80	0.817
ABC'	0.40	0.230	0.83	0.688	0.75	0.476
$AB'C''$	0.32	0.226	0.86	0.691	0.70	0.476
ABX	0.24	0.226	0.76	0.690	0.50	0.476
$AB'X$	0.20	0.224	0.80	0.690	0.47	0.476
AXY	0.08	0.219	0.55	0.694	0.34	0.476
XYZ	0.10	0.216	0.35	0.693	0.23	0.476
AB	0.68	0.678	0.85	0.849	0.77	0.760
AB'	0.44	0.274	0.77	0.560	0.70	0.477
AX	0.28	0.270	0.62	0.560	0.45	0.477
XY	0.12	0.266	0.32	0.561	0.22	0.477
A	0.60	0.594	0.72	0.804	0.72	0.679
X	0.32	0.307	0.26	0.583	0.32	0.468

[a] Parameter values for this fit are $\alpha = 0.98$, $N = 456$, $\gamma = 0.117$, $\omega = 0.183$, $t = 0.466$, $d = 0.007$, $s = 0.014$, $p = 0.746$, $q = 0.010$, $w = 0.982$, Crit $= 0.071$.

[b] Parameter values for this fit are $\alpha = 0.97$, $N = 381$, $\gamma = 0.199$, $\omega = 0.153$, $t = 0.029$, $d = 0.022$, $s = 0.038$, $p = 0.010$, $q = 0.916$, $w = 0.989$, Crit $= -0.045$.

[c] Parameter values for this fit are $\alpha = 0.97$, $N = 483$, $\gamma = 0.130$, $\omega = 0.116$, $t = 0.013$, $d = 0.012$, $s = 0.550$, $p = 0.012$, $q = 0.914$, $w = 0.935$, Crit $= 0.012$.

phase. For the "intact" rule, subjects were instructed to say "old" only if all words in the test probe had been studied in the same triplet (i.e., for the ABC, AB, and A probes). For the "any-old" decision rule, subjects were to say "old" if any word in the test probe was from the list, even if other words in the probe were new (i.e., for all but the XYZ, XY, and X probes). For the "all-old" rule, subjects were to say "old" if all words in the probe were from the study list, but not necessarily from the same triplet (i.e., for the ABC, ABC', $AB'C''$, AB, AB', and A probes). For each decision rule, the frequency of probe types presented was such that there was a roughly equal number of "new" and "old" trials.

The data points reported in Table 9 as P_{obs} come from Clark and Shiffrin's Figs. 1 to 3, and are the probability of classifying a given probe type as "old" averaged over 128 subjects. The 99% confidence intervals for these point estimates as read from the figures range from approximately ± 0.05 to ± 0.10.

To model these data with TODAM, one must make assumptions about the encoding of the triplets at time of study and about the encoding and utilization of

the probes at time of test as a function of the decision rule instructions. The assumptions made here are just one of several sets of plausible assumptions and are intended only to illustrate the basic procedure. More sophisticated assumptions could probably improve the fit of the model which, as shown in Table 9, is not particularly good. The major intent of this paper, as stated in the introduction, is to facilitate application of the model by providing the basic building blocks for model fits so that researchers can concentrate on such higher-level problems as the optimal set of encoding assumptions.

With this proviso, the assumptions made to obtain the fits reported in Table 9 were as follows. Encoding of the jth triplet presented during the study phase was modeled as

$$\mathbf{M}_j = \alpha\mathbf{M}_{j-1} + \gamma\mathbf{A} + \gamma\mathbf{B} + \gamma\mathbf{C} + \omega(\mathbf{A} * \mathbf{B}) + \omega(\mathbf{B} * \mathbf{C}) + \psi(\mathbf{A} * \mathbf{B} * \mathbf{C}), \qquad (13)$$

where $3\gamma + 2\omega + \psi = 1$. Encoding during study was modeled identically for the three decision rule conditions, since subjects were not told which decision rule to use until afterwards. For the test probes, it was assumed that subjects utilized all available information (i.e., triplet, pair, and single-item information for triplet probes; pair and single-item information for pair probes; and single-item information for single-item probes), but to varying degrees as a function of the decision rule they were instructed to use. Thus test triplet ABC, pair AB, and singleton A encoding was modeled as

$$\begin{gathered} t(\mathbf{A} * \mathbf{B} * \mathbf{C}) + d(\mathbf{A} * \mathbf{B}) + d(\mathbf{B} * \mathbf{C}) + s\mathbf{A} + s\mathbf{B} + s\mathbf{C}, \\ p(\mathbf{A} * \mathbf{B}) + q\mathbf{A} + q\mathbf{B}, \end{gathered} \qquad (14)$$

and

$$w\mathbf{A},$$

respectively. Parameters t, d, s, p, q, and w were free to vary between 0 and 1 and were assumed to be different for the three decision rule conditions to reflect differences in diagnosticity of a particular information component under the three rules. (A high memory trace match for component $(\mathbf{A} * \mathbf{B} * \mathbf{C})$, for example, should contribute more to an "old" decision under the "intact" rule where, in fact, it is the only decisive piece of information than under the "any-old" or "all-old" rule.)

The distributions of dot-product resemblance between the probes and the memory vector are then obtained by expressing the probe as in (14) and the memory vector as in (13), taking into consideration the number of triplets in the memory vector that will give a partial or complete match with the probe. For probes ABC, ABX, AXY, AB, AX, and A, the memory vector $\mathbf{M}$ consists of 1 triplet that will (partially) match the probe and 20 remaining triplets that are complete mismatches, i.e.,

$$\mathbf{M} = c_j(ABC) + k(DEF), \qquad (15)$$

where $c_j = \alpha^{m-j}$, $k = \sum_{j=1}^{m} \alpha^{m-j} - c_j$, with j denoting the serial position of the matching study triplet and m the length of the study list. (Since the data reported

were averaged over serial position, modeling was done with an average c_j.) For probes ABC', $AB'X$, and AB', the memory vector consists of two partially matching triplets, i.e.,

$$\mathbf{M} = c_j(ABC) + c_l(A'B'C') + k'(DEF), \tag{16}$$

where

$$c_j = \alpha^{m-j},\ c_l = \alpha^{m-l}\ (l \neq j), \text{ and } k' = \sum_{j=1}^{m} \alpha^{m-j} - c_j - c_l.$$

Finally, for probes of type $AB'C''$, the memory vector consists of three partially matching triplets, i.e.,

$$\mathbf{M} = c_j(ABC) + c_l(A'B'C') + c_n(A''B''C'') + k''(DEF), \tag{17}$$

where

$$c_j = \alpha^{m-j},\ c_l = \alpha^{m-l},\ c_n = \alpha^{m-n}\ (l \neq j \neq n),$$
$$k'' = \sum_{j=1}^{m} \alpha^{m-j} - c_j - c_l - c_n.$$

The expectations and variances of the dot-product resemblance distributions between the 13 probe types and the memory vector as a function of the parameters α, N, γ, ω, t, d, s, p, q, and w are listed in Appendix C. These expressions were obtained by crossing the appropriate expansion of the probe (as in (14)) with the appropriate expansion of the memory vector (as in (15) to (17)). Since the components of either expansion are assumed to be mutually independent under TODAM, the expectation and variance of the sum of the component matches can be expressed as the sum of the component matches expectations and variances, respectively. The component match variances can, of course, be found in Tables 1 to 6.

Finally, a criterion cutoff level for resemblance which was assumed to be different for each decision rule was used to predict the probability of an "old" classification for each probe type.

The model fits are summarized in Table 9. As already mentioned, the fits are not particularly good (accounting for only 78.3, 24.3, and 36.7% of the variance for the three decision rule conditions, respectively), their major deficiency being the model's inability to discriminate between incompletely matching probes (e.g., between probes ABC', $AB'C''$, ABX, $AB'X$, AXY, and XYZ) for all decision rules. This is the result of these probes having an expected value of zero for their dot-product resemblance, so that differences in performance must be modeled by differences in dot-product resemblance variances. This severely restricts the range of differences between such probes that the model is capable of predicting without further modifications. (For example, the model could not predict $P(\text{old}) < 0.5$ for one probe and $P(\text{old}) > 0.5$ for another probe, given a common value of Crit, no matter how different the variances of the resemblance distributions of those two probes (assuming unimodal and symmetric resemblance distributions).) The value of fits like the one reported here is precisely to point out the need for modifications and refinements of the basic model.

Conclusion

The three sample applications of TODAM using the expectation and variance expressions derived in this paper are intended to illustrate basic model application procedures for situations with partial stimulus encoding during recall or retrieval and for complex associations. Other researchers are encouraged to find ways to improve on the fits reported here. An important lesson is that the distributed memory assumption by itself determines the fit of the model only partially. Additional assumptions about how subjects interpret and thus encode stimuli, or about the way various pieces of information are combined in a post-memory decision stage, are necessary to model most data sets, and model fits can be as different as those assumptions.

APPENDIX A

Derivations of Resemblance Distribution Variances Reported in Tables 1 to 6

Ia. *Table* 1a

(i) $\mathrm{Var}(\mathbf{f}\cdot\mathbf{f}) = N\,\mathrm{Var}(\mathbf{X}^2)$

(ii) $\mathrm{Var}(\mathbf{g}\cdot\mathbf{f}) = N\,\mathrm{Var}(\mathbf{XY})$.

Ib. *Table* 1b

(i) $\mathrm{Var}(p\mathbf{f}\cdot\mathbf{f}) = pN\,\mathrm{Var}(\mathbf{X}^2)$

(ii) $\mathrm{Var}(p\mathbf{g}\cdot\mathbf{f}) = pN\,\mathrm{Var}(\mathbf{XY})$.

IIa. *Table* 2a

(i) $\mathrm{Var}[\mathbf{f}\cdot(\mathbf{f}*\mathbf{f})]$

$$= \mathrm{Var}(\mathbf{X}^3) + \left(2\,\mathrm{Int}\left(\frac{N}{4}\right)\right)\mathrm{Var}(\mathbf{X}^2\mathbf{Y}) + (N-1)\,\mathrm{Var}(2\mathbf{X}^2\mathbf{Y})$$

$$+\left[\sum_{i=1}^{(N-1)/2} i + \sum_{i=1}^{(N-3)/2} i + 2\sum_{i=1}^{(N-5)/2} \mathrm{Int}\left(\frac{i+1}{2}\right)\right]\mathrm{Var}(2\mathbf{XYZ})$$

$$+\ (N-1)(N-2)\,\mathrm{Cov}(2\mathbf{X}^2\mathbf{Y}, 2\mathbf{Z}^2\mathbf{Y}) + 2(N-1)\,\mathrm{Cov}(\mathbf{Z}^3, 2\mathbf{X}^2\mathbf{Z})$$

(ii) $\mathrm{Var}[\mathbf{h}\cdot(\mathbf{f}*\mathbf{f})]$

$$= \left(2\,\mathrm{Int}\left(\frac{N}{4}\right)+1\right)\mathrm{Var}(\mathbf{X}^2\mathbf{Y})$$

$$+\left[3\,\frac{N-1}{2} + 4\sum_{i=1}^{(N-3)/2} i - \sum_{i=1}^{\mathrm{Int}((N-3)/4)} i - \sum_{i=1}^{\mathrm{Int}((N-1)/4)} i\right]\mathrm{Var}(2\mathbf{XYZ})$$

(iii) $\mathrm{Var}[\mathbf{f}\cdot(\mathbf{f}*\mathbf{g})]$

$$= N\,\mathrm{Var}(\mathbf{X}^2\mathbf{Y}) + \left(\sum_{i=(N-1)/2}^{N-1} i + \sum_{i=(N-1)/2}^{N-2} i\right)\mathrm{Var}(\mathbf{XYZ})$$
$$+ 2\binom{N}{2}\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}, \mathbf{Z}^2\mathbf{Y})$$

(iv) $\mathrm{Var}[\mathbf{h}\cdot(\mathbf{f}*\mathbf{g}) = \left(N^2 - 2\sum_{i=1}^{(N-1)/2} i\right)\mathrm{Var}(\mathbf{XYZ}).$

IIb. *Table* 2b

(i) $\mathrm{Var}[p\mathbf{f}\cdot\mathbf{f}*\mathbf{f})]$

$$= p\,\mathrm{Var}(\mathbf{X}^3) + \left(2p\,\mathrm{Int}\left(\frac{N}{4}\right)\right)\mathrm{Var}(\mathbf{X}^2\mathbf{Y}) + p(N-1)\,\mathrm{Var}(2\mathbf{X}^2\mathbf{Y})$$
$$+ p\left[\sum_{i=1}^{(N-1)/2} i + \sum_{i=1}^{(N-3)/2} i + 2\sum_{i=1}^{(N-5)/2}\mathrm{Int}\left(\frac{i+1}{2}\right)\right]\mathrm{Var}(2\mathbf{XYZ})$$
$$+ p^2(N-1)(N-2)\,\mathrm{Cov}(2\mathbf{X}^2\mathbf{Y}, 2\mathbf{Z}^2\mathbf{Y}) + 2p^2(N-1)\,\mathrm{Cov}(\mathbf{Z}^3, 2\mathbf{X}^2\mathbf{Z})$$

(ii) $\mathrm{Var}[p\mathbf{h}\cdot(\mathbf{f}*\mathbf{f})] = p\{\mathrm{Var}[\mathbf{h}\cdot(\mathbf{f}*\mathbf{f})]\}$

(iii) $\mathrm{Var}[p\mathbf{f}\cdot(\mathbf{f}*\mathbf{g})]$

$$= pN\,\mathrm{Var}(\mathbf{X}^2\mathbf{Y}) + p\left(\sum_{i=(N-1)/2}^{N-1} i + \sum_{i=(N-1)/2}^{N-2} i\right)\mathrm{Var}(\mathbf{XYZ})$$
$$+ 2\binom{pN}{2}\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}, \mathbf{Z}^2\mathbf{Y})$$

(iv) $\mathrm{Var}[p\mathbf{h}\cdot(\mathbf{f}*\mathbf{g})] = p\{\mathrm{Var}[\mathbf{h}\cdot(\mathbf{f}*\mathbf{g})]\}.$

IIIa. *Table* 3a

(i) $\mathrm{Var}[(\mathbf{f}*\mathbf{f})\cdot(\mathbf{f}*\mathbf{f})]$

$$= N\,\mathrm{Var}(\mathbf{X}^4) + \frac{N(N-1)}{2}\mathrm{Var}(4\mathbf{X}^2\mathbf{Y}^2)$$
$$+ \left(\frac{N-1}{2}\right)^2\mathrm{Var}(4\mathbf{X}^2\mathbf{YZ}) + \frac{(N-1)(N-3)(2N-1)}{24}\mathrm{Var}(8\mathbf{WXYZ})$$
$$+ 2N(N-1)\,\mathrm{Cov}(\mathbf{X}^4, 4\mathbf{X}^2\mathbf{Y}^2) + 2(N-2)\binom{N}{2}\mathrm{Cov}(4\mathbf{X}^2\mathbf{Y}^2, 4\mathbf{X}^2\mathbf{Z}^2)$$

(ii) $\mathrm{Var}[(\mathbf{f} * \mathbf{h}) \cdot (\mathbf{f} * \mathbf{f})]$

$$= N\,\mathrm{Var}(\mathbf{X}^3\mathbf{Y}) + \left(6 \sum_{i=1}^{(N-1)/2} i + 2 \sum_{i=1}^{(N-3)/2} i\right) \mathrm{Var}(2\mathbf{X}^2\mathbf{YZ})$$
$$+ 2\left(\sum_{i=1}^{(N-1)/2} i + \sum_{i=1}^{(N-3)/2} i\right) \mathrm{Var}(\mathbf{X}^2\mathbf{YZ})$$
$$+ \left\{8\left[\sum_{i=1}^{(N-1)/2} \binom{i}{2}\right] + \sum_{i=1}^{(N-1)/2} i(2i-1) + \sum_{i=1}^{(N-3)/2} i(2i-1)\right\}$$
$$\times \mathrm{Var}(2\mathbf{WXYZ})$$
$$+ 2N\binom{N-1}{2} \mathrm{Cov}(2\mathbf{X}^2\mathbf{YZ}, 2\mathbf{W}^2\mathbf{YZ})$$

(iii) $\mathrm{Var}[(\mathbf{g} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{f})]$

$$= N\,\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2) + 2\left(\sum_{i=1}^{(N-1)/2} i + \sum_{i=1}^{(N-3)/2} i\right) \mathrm{Var}(2\mathbf{X}^2\mathbf{YZ})$$
$$+ \left(3 \sum_{i=1}^{(N-1)/2} i^2 + \sum_{i=1}^{(N-3)/2} i^2\right) \mathrm{Var}(4\mathbf{WXYZ})$$

(iv) $\mathrm{Var}[(\mathbf{h} * \mathbf{j}) \cdot (\mathbf{f} * \mathbf{f})]$

$$= \left[\sum_{i=1}^{(N+1)/2} (2i-1) + \sum_{i=1}^{(N-1)/2} (2i-1)\right] \mathrm{Var}(\mathbf{X}^2\mathbf{YZ})$$
$$+ \left[4 \sum_{i=1}^{(N-1)/2} i^2 + \sum_{i=1}^{(N+1)/2} (2i-1)(i-1)\right.$$
$$\left. + \sum_{i=1}^{(N-1)/2} (2i-1)(i-1)\right] \mathrm{Var}(2\mathbf{WXYZ})$$

(v) $\mathrm{Var}[(\mathbf{f} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{g})]$

$$= N^2\,\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2) + \left(\sum_{i=1}^{N-1} i + 2 \sum_{k=1}^{N-2} \sum_{i=1}^{k} i\right) \mathrm{Var}(2\mathbf{WXYZ})$$
$$4N\binom{N}{2} \mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2, \mathbf{X}^2\mathbf{Z}^2)$$

(vi) $\mathrm{Var}[(\mathbf{f} * \mathbf{h}) \cdot (\mathbf{f} * \mathbf{g})]$

$$= N^2\,\mathrm{Var}(\mathbf{X}^2\mathbf{YZ}) + 2\left[\sum_{i=1}^{N} \binom{i}{2} + \sum_{i=1}^{N-1} \binom{i}{2}\right] \mathrm{Var}(\mathbf{WXYZ})$$
$$+ (N^3 - N^2)\,\mathrm{Cov}(\mathbf{X}^2\mathbf{YZ}, \mathbf{W}^2\mathbf{YZ})$$

(vii) $\mathrm{Var}[(\mathbf{h} * \mathbf{j}) \cdot (\mathbf{f} * \mathbf{g})]$

$$= \left(\sum_{i=2}^{N} i^2 + \sum_{i=1}^{N-1} i^2 \right) \mathrm{Var}(\mathbf{WXYZ}).$$

IIIb. *Table* 3b

(i) $\mathrm{Var}[(p\mathbf{f} * \mathbf{f}) \cdot (\mathbf{f} * \mathbf{f})]$

$$= pN\,\mathrm{Var}(\mathbf{X}^4) + \binom{pN}{2} \mathrm{Var}(4\mathbf{X}^2\mathbf{Y}^2)$$

$$+ (pN^2 - p^2N^2)\,\mathrm{Var}(2\mathbf{X}^2\mathbf{Y}^2) + p\left(\frac{pN-1}{2}\right)^2 \mathrm{Var}(4\mathbf{X}^2\mathbf{YZ})$$

$$+ p\,\frac{pN^2 - p^2N^2}{4}\,\mathrm{Var}(3\mathbf{X}^2\mathbf{YZ}) + \left\{(1-p)\left(\frac{pN-1}{2}\right)^2\right.$$

$$\left. + p\left[\left(\frac{N-1}{2}\right)^2 - \left(\frac{pN-1}{2}\right)^2\right]\right\} \mathrm{Var}(2\mathbf{X}^2\mathbf{YZ})$$

$$+ (1-p)\left(\frac{pN-1}{2}\right)\left(\frac{N-pN-1}{2}\right) \mathrm{Var}(\mathbf{X}^2\mathbf{YZ})$$

$$+ p^4\,\frac{(N-1)(N-3)(2N-1)}{24}\,\mathrm{Var}(8\mathbf{WXYZ})$$

$$+ 4(p^3 - p^4)\,\frac{(N-1)(N-3)(2N-1)}{24}\,\mathrm{Var}(6\mathbf{WXYZ})$$

$$+ (6p^4 - 12p^3 + 6p^2)\,\frac{(N-1)(N-3)(2N-1)}{24}\,\mathrm{Var}(4\mathbf{WXYZ})$$

$$+ (4p - 12p^2 + 12p^3 - 4p^4)\,\frac{(N-1)(N-3)(2N-1)}{24}\,\mathrm{Var}(2\mathbf{WXYZ})$$

$$+ 2pN(pN-1)\,\mathrm{Cov}(\mathbf{X}^4, 4\mathbf{X}^2\mathbf{Y}^2) + 2pN(N-pN)\,\mathrm{Cov}(\mathbf{X}^4, 2\mathbf{X}^2\mathbf{Y}^2)$$

$$+ 2(pN-2)\binom{pN}{2}\,\mathrm{Cov}(4\mathbf{X}^2\mathbf{Y}^2, 4\mathbf{X}^2\mathbf{Z}^2)$$

$$+ pN(N-pN)(N-pN-1)\,\mathrm{Cov}(2\mathbf{X}^2\mathbf{Y}^2, 2\mathbf{X}^2\mathbf{Z}^2)$$

$$+ 2(N-pN)\binom{pN}{2}\,\mathrm{Cov}(4\mathbf{X}^2\mathbf{Y}^2, 2\mathbf{X}^2\mathbf{Z}^2)$$

(ii) $\mathrm{Var}[(p\mathbf{f} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{f})]$

$$= pN\,\mathrm{Var}(\mathbf{X}^3\mathbf{Y}) + p\left(6\sum_{i=1}^{(N-1)/2} i + 2\sum_{i=1}^{(N-3)/2} i\right)\mathrm{Var}(2\mathbf{X}^2\mathbf{YZ})$$

$$+ 2p\left(\sum_{i=1}^{(N-1)/2} i + 2\sum_{i=1}^{(N-3)/2} i\right)\mathrm{Var}(\mathbf{X}^2\mathbf{YZ})$$

$$+ p\left\{8\sum_{i=1}^{(N-1)/2}\binom{i}{2} + \sum_{i=1}^{(N-1)/2} i(2i-1) + \sum_{i=1}^{(N-3)/2} i(2i-1)\right\}\mathrm{Var}(2\mathbf{WXYZ})$$

$$+ 2N\binom{pN-1}{2}\mathrm{Cov}(2\mathbf{X}^2\mathbf{YZ}, 2\mathbf{W}^2\mathbf{YZ})$$

(iii) $\mathrm{Var}[(\mathbf{f} * p\mathbf{g}) \cdot (\mathbf{f} * \mathbf{f})] = p\{\mathrm{Var}[(\mathbf{f} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{f})]\}$

(iv) $\mathrm{Var}[(p\mathbf{g} * \mathbf{g}) \cdot (\mathbf{f} * \mathbf{f})] = pN\,\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2)$

$$+ (p + p^2)\left(\sum_{i=1}^{(N-1)/2} i + \sum_{i=1}^{(N-3)/2} i\right)\mathrm{Var}(2\mathbf{X}^2\mathbf{YZ})$$

$$+ p(1-p)\left(\sum_{i=1}^{(N-1)/2} i + \sum_{i=1}^{(N-3)/2} i\right)\mathrm{Var}(\mathbf{X}^2\mathbf{YZ})$$

$$+ p^2\left(3\sum_{i=1}^{(N-1)/2} i^2 + \sum_{i=1}^{(N-3)/2} i^2\right)\mathrm{Var}(4\mathbf{WXYZ})$$

$$+ p(1-p)\left(3\sum_{i=1}^{(N-1)/2} i^2 + \sum_{i=1}^{(N-3)/2} i^2\right)\mathrm{Var}(2\mathbf{WXYZ})$$

(v) $\mathrm{Var}[(p\mathbf{h} * \mathbf{j}) \cdot (\mathbf{f} * \mathbf{f})] = p\{\mathrm{Var}[(\mathbf{h} * \mathbf{j}) \cdot (\mathbf{f} * \mathbf{f})]\}$

(vi) $\mathrm{Var}[(p\mathbf{f} * \mathbf{f}) \cdot (\mathbf{f} * \mathbf{g})]$

$$= pN\,\mathrm{Var}(\mathbf{X}^3\mathbf{Y}) + p^2\left(6\sum_{i=1}^{(N-1)/2} i + 2\sum_{i=1}^{(N-3)/2} i\right)\mathrm{Var}(2\mathbf{X}^2\mathbf{YZ})$$

$$+ \left[2p(1-p)\left(6\sum_{i=1}^{(N-1)/2} i + 2\sum_{i=1}^{(N-3)/2} i\right)\right.$$

$$\left. + 2p\left(\sum_{i=1}^{(N-1)/2} i + \sum_{i=1}^{(N-3)/2} i\right)\right]\mathrm{Var}(\mathbf{X}^2\mathbf{YZ}) + p^2\left[4\sum_{i=1}^{(N-1)/2} i(i-1)\right.$$

$$\left. + \sum_{i=1}^{(N-1)/2} i(2i-1) + \sum_{i=1}^{(N-3)/2} i(2i-1)\right]\mathrm{Var}(2\mathbf{WXYZ})$$

$$+ 2p(1-p)\left[4\sum_{i=1}^{(N-1)/2} i(i-1) + \sum_{i=1}^{(N-1)/2} i(2i-1)\right.$$

$$+\sum_{i=1}^{(N-3)/2} i(2i-1)\Bigg]\operatorname{Var}(\mathbf{WXYZ})$$

$$+2\frac{pN(pN-1)(N-2)}{2}\operatorname{Cov}(2\mathbf{Z}^2\mathbf{XY}, 2\mathbf{W}^2\mathbf{XY})$$

$$+2\frac{(N-pN)(N-pN-1)(N-2)}{2}\operatorname{Cov}(\mathbf{Z}^2\mathbf{XY}, \mathbf{W}^2\mathbf{XY})$$

$$+2\frac{pN(N-pN)(N-2)}{2}\operatorname{Cov}(2\mathbf{Z}^2\mathbf{XY}, \mathbf{W}^2\mathbf{XY})$$

(vii) $\operatorname{Var}[(p\mathbf{f}*\mathbf{g})\cdot(\mathbf{f}*\mathbf{g})]$

$$= pN^2\operatorname{Var}(\mathbf{X}^2\mathbf{Y}^2)+p^2\left(\frac{2N^3-3N^2+N}{6}\right)\operatorname{Var}(2\mathbf{WXYZ})$$

$$+2p(1-p)\left(\frac{2N^3-3N^2+N}{6}\right)\operatorname{Var}(\mathbf{WXYZ})$$

$$+2\left[pN\binom{N}{2}+N\binom{pN}{2}\right]\operatorname{Cov}(\mathbf{X}^2\mathbf{Y}^2, \mathbf{X}^2\mathbf{Z}^2)$$

(viii) $\operatorname{Var}[(p\mathbf{f}*\mathbf{h})\cdot(\mathbf{f}*\mathbf{g})]$

$$= pN^2\operatorname{Var}(\mathbf{X}^2\mathbf{YZ})+2p\left[\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right]\operatorname{Var}(\mathbf{WXYZ})$$

$$+2N\binom{pN}{2}\operatorname{Cov}(\mathbf{X}^2\mathbf{YZ}, \mathbf{T}^2\mathbf{YZ})$$

(ix) $\operatorname{Var}[(\mathbf{f}*p\mathbf{h})\cdot(\mathbf{f}*\mathbf{g})] = p\{\operatorname{Var}[(\mathbf{f}*\mathbf{h})\cdot(\mathbf{f}*\mathbf{g})]\}$

(x) $\operatorname{Var}[(p\mathbf{h}*\mathbf{j})\cdot(\mathbf{f}*\mathbf{g})] = p\{\operatorname{Var}[(\mathbf{h}*\mathbf{j})\cdot(\mathbf{f}*\mathbf{g})]\}$

IIIc. *Table* 3c

(i) $\operatorname{Var}[(\mathbf{f}*\mathbf{f})\cdot(p\mathbf{f}*p\mathbf{f})]$

$$= pN\operatorname{Var}(\mathbf{X}^4)+\binom{pN}{2}\operatorname{Var}(4\mathbf{X}^2\mathbf{Y}^2)$$

$$+(pN^2-p^2N^2)\operatorname{Var}(2\mathbf{X}^2\mathbf{Y}^2)+p\left(\frac{pN-1}{2}\right)^2\operatorname{Var}(4\mathbf{X}^2\mathbf{YZ})$$

$$+(1-p)\left(\frac{pN-1}{2}\right)^2\operatorname{Var}(2\mathbf{X}^2\mathbf{YZ})$$

$$+\frac{p^4(N-1)(N-3)(2N-1)}{24}\operatorname{Var}(8\mathbf{WXYZ})$$

$$+\frac{p^2(p-1)^2(N-1)(N-3)(2N-1)}{12}\operatorname{Var}(4\mathbf{WXYZ})$$

$$+2pN(pN-1)\operatorname{Cov}(\mathbf{X}^4, 4\mathbf{X}^2\mathbf{Y}^2)+2pN(N-pN)\operatorname{Cov}(\mathbf{X}^4, 2\mathbf{X}^2\mathbf{Y}^2)$$

$$+2(pN-2)\binom{pN}{2}\operatorname{Cov}(4\mathbf{X}^2\mathbf{Y}^2, 4\mathbf{X}^2\mathbf{Z}^2)$$

$$+pN(N-pN)(N-pN-1)\operatorname{Cov}(2\mathbf{X}^2\mathbf{Y}^2, 2\mathbf{X}^2\mathbf{Z}^2)$$

$$+2(N-pN)\binom{pN}{2}\operatorname{Cov}(4\mathbf{X}^2\mathbf{Y}^2, 2\mathbf{X}^2\mathbf{Z}^2)$$

(ii) $\operatorname{Var}[(\mathbf{f}*\mathbf{g})\cdot(p\mathbf{f}*p\mathbf{f})]$

$$=pN\operatorname{Var}(\mathbf{X}^3\mathbf{Y})+p^2\left(6\sum_{i=1}^{(N-1)/2} i+2\sum_{i=1}^{(N-3)/2} i\right)\operatorname{Var}(2\mathbf{X}^2\mathbf{YZ})$$

$$+2p\left(\sum_{i=1}^{(N-1)/2} i+\sum_{i=1}^{(N-3)/2} i\right)\operatorname{Var}(\mathbf{X}^2\mathbf{YZ})$$

$$+p^2\left[4\sum_{i=1}^{(N-1)/2} i(i-1)+\sum_{i=1}^{(N-1)/2} i(2i-1)\right.$$

$$\left.+\sum_{i=1}^{(N-3)/2} i(2i-1)\right]\operatorname{Var}(2\mathbf{WXYZ})$$

$$+pN(pN-1)(N-2)\operatorname{Cov}(2\mathbf{Z}^2\mathbf{XY}, 2\mathbf{W}^2\mathbf{XY})$$

(iii) $\operatorname{Var}[(\mathbf{g}*\mathbf{g})\cdot(p\mathbf{f}*p\mathbf{f})]$

$$=pN\operatorname{Var}(\mathbf{X}^2\mathbf{Y}^2)+\left[(p+p^2)\left(\sum_{i=1}^{(N-1)/2} i+\sum_{i=1}^{(N-3)/2} i\right)\right]\operatorname{Var}(2\mathbf{X}^2\mathbf{YZ})$$

$$+p^2\left(3\sum_{i=1}^{(N-1)/2} i^2+\sum_{i=1}^{(N-3)/2} i^2\right)\operatorname{Var}(4\mathbf{WXYZ})$$

(iv) $\operatorname{Var}[(\mathbf{g}*\mathbf{h})\cdot(p\mathbf{f}*p\mathbf{f})]$

$$=p\left[\sum_{i=1}^{(N+1)/2}(2i-1)+\sum_{i=1}^{(N-1)/2}(2i-1)\right]\operatorname{Var}(\mathbf{X}^2\mathbf{YZ})$$

$$+p^2\left[4\sum_{i=1}^{(N-1)/2} i^2+\sum_{i=1}^{(N+1)/2}(2i-1)(i-1)\right.$$

$$\left.+\sum_{i=1}^{(N-1)/2}(2i-1)(i-1)\right]\operatorname{Var}(2\mathbf{WXYZ}).$$

IVa. *Table* 4a

(i) $\mathrm{Var}[\mathbf{f}\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$= N^2\,\mathrm{Var}(\mathbf{X}^2\mathbf{YZ}) + \frac{2N^3-3N^2+N}{3}\,\mathrm{Var}(\mathbf{WXYZ})$$

$$+ 2N\binom{N}{2}\,\mathrm{Cov}(\mathbf{X}^2\mathbf{YZ}, \mathbf{W}^2\mathbf{YZ})$$

(ii) $\mathrm{Var}[\mathbf{j}\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$= \left\{2\left[\sum_{i=1}^{N} i + \sum_{i=1}^{(N-3)/2}\left(\sum_{j=1}^{N} j - \sum_{k=1}^{i} k + \sum_{l=N-i}^{N-1} l\right)\right]\right.$$

$$\left. + \left(\sum_{j=1}^{N} j - \sum_{k=1}^{(N-1)/2} k + \sum_{l=(N+1)/2}^{N-1} l\right)\right\}\mathrm{Var}(\mathbf{WXYZ}).$$

IVb. *Table* 4b

(i) $\mathrm{Var}[p\mathbf{f}\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$= pN^2\,\mathrm{Var}(\mathbf{X}^2\mathbf{YZ}) + p\,\frac{(2N^3-3N^2+N)}{3}\,\mathrm{Var}(\mathbf{WXYZ})$$

$$+ 2N\binom{pN}{2}\,\mathrm{Cov}(\mathbf{X}^2\mathbf{YZ}, \mathbf{W}^2\mathbf{YZ})$$

(ii) $\mathrm{Var}[p\mathbf{j}\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})] = p\{\mathrm{Var}[\mathbf{j}\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]\}$

Va. *Table* 5a

(i) $\mathrm{Var}[((\mathbf{f}*\mathbf{g})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$= N^2\,\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}) + 2N\left[2\sum_{i=(N-1)/2}^{(N-2)} i + (N-1)\right]\mathrm{Var}(\mathbf{X}^2\mathbf{WYZ})$$

$$+ \frac{N^2-N}{2}\,\mathrm{Var}(2\mathbf{VWXYZ})$$

$$+ \frac{1}{192}(115N^4+288N^3+50N^2+96N+27)\,\mathrm{Var}(\mathbf{VWXYZ})$$

$$+ 4N\binom{N}{2}\,\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}, \mathbf{X}^2\mathbf{W}^2\mathbf{Z}) + 4\binom{N}{2}^2\,\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}, \mathbf{V}^2\mathbf{W}^2\mathbf{Z})$$

$$+ 2\left[2\left(\sum_{i=(N-1)/2}^{N-1} i + \sum_{i=(N-1)/2}^{N-2} i\right)\binom{N}{2}\right]\mathrm{Cov}(\mathbf{X}^2\mathbf{WYZ}, \mathbf{T}^2\mathbf{WYZ})$$

(ii) $\mathrm{Var}[(\mathbf{f}*\mathbf{f})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$=\left(2\sum_{i=(N+1)/2}^{N-1} i+N\right)\mathrm{Var}(\mathbf{X}^3\mathbf{YZ})+\left[2\left(\sum_{j=1}^{(N+1)/4}\sum_{i=1}^{((N-3)/2)+2j} i\right.\right.$$

$$\left.\left.+\sum_{j=1}^{(N-3)/4}\left(\sum_{i=1}^{N} i+N+2-4j\right)\right)\right]\mathrm{Var}(\mathbf{X}^2\mathbf{WYZ})$$

$$+(N-1)\left(2\sum_{i=(N+1)/2}^{N-1} i+N\right)\mathrm{Var}(2\mathbf{X}^2\mathbf{WYZ})$$

$$+\frac{1}{192}(57.5N^4+182N^3-47N^2+26N+217.5)\,\mathrm{Var}(2\mathbf{VWXYZ})$$

(iii) $\mathrm{Var}[(\mathbf{f}*\mathbf{j})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$=N\left(2\sum_{i=(N+1)/2}^{N-1} i+N\right)\mathrm{Var}(\mathbf{X}^2\mathbf{WYZ})$$

$$+\frac{1}{192}(115N^4-144N^3+50N^2-48N+27)\,\mathrm{Var}(\mathbf{VWXYZ})$$

$$+2\left(\frac{3N^2+1}{4}\right)\binom{N}{2}\mathrm{Cov}(\mathbf{X}^2\mathbf{WYZ},\mathbf{V}^2\mathbf{WYZ})$$

(iv) $\mathrm{Var}[(\mathbf{j}*\mathbf{k})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$=\left\{2\sum_{i=1}^{(N+1)/2} i\sum_{j=1}^{((N-1)/2)+i} j+2\sum_{i=1}^{(N-3)/2}\left[\left(i+\frac{N+1}{2}\right)\right.\right.$$

$$\left.\times\left(\sum_{j=1}^{N} j-\sum_{k=1}^{i} k+\sum_{l=N-i}^{N-1} l\right)\right]$$

$$\left.+N\left(\sum_{j=1}^{N} j-\sum_{k=1}^{(N-1)/2} k+\sum_{l=(N+1)/2}^{N-1} l\right)\right\}\mathrm{Var}(\mathbf{VWXYZ}).$$

Vb. *Table* 5b

(i) $\mathrm{Var}[(p\mathbf{f}*\mathbf{g})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$=pN^2\,\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z})+2pN\left[2\sum_{i=(N-1)/2}^{N-2} i+(N-1)\right]\mathrm{Var}(\mathbf{X}^2\mathbf{WYZ})$$

$$+\frac{pN^2-pN}{2}\mathrm{Var}(2\mathbf{VWXYZ})+\left[\frac{p}{192}(115N^4+50N^2+27)\right.$$

$$\left.-pN^2-\frac{p(3N^3-4N^2+N)}{2}-(pN^2-pN)\right]\mathrm{Var}(\mathbf{VWXYZ})$$

$$+2\left[pN\binom{N}{2}+N\binom{pN}{2}\right]\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z},\mathbf{X}^2\mathbf{W}^2\mathbf{Z})$$

$$+2N(N-1)\binom{pN}{2}\operatorname{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}, \mathbf{V}^2\mathbf{W}^2\mathbf{Z})$$

$$+2\left(2\sum_{i=(N-1)/2}^{N-2} i+(N-1)\right)\left[\binom{pN}{2}+p\binom{N}{2}\right]$$

$$\times\operatorname{Cov}(\mathbf{X}^2\mathbf{WYZ}, \mathbf{T}^2\mathbf{WYZ})$$

(ii) $\operatorname{Var}[(p\mathbf{f}*\mathbf{f})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})] = p\{\operatorname{Var}[(\mathbf{f}*\mathbf{f})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]\}$

(iii) $\operatorname{Var}[(p\mathbf{f}*\mathbf{j})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$= pN\left(2\sum_{i=(N+1)/2}^{N-1} i+N\right)\operatorname{Var}(\mathbf{X}^2\mathbf{WYZ})$$

$$+\frac{p}{192}(115N^4-144N^3+50N^2-48N+27)\operatorname{Var}(\mathbf{VWXYZ})$$

$$+2\left(\frac{3N^4+1}{4}\right)\binom{pN}{2}\operatorname{Cov}(\mathbf{X}^2\mathbf{WYZ}, \mathbf{V}^2\mathbf{WYZ})$$

(iv) $\operatorname{Var}[(\mathbf{f}*p\mathbf{j})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})] = p\{\operatorname{Var}[(\mathbf{f}*\mathbf{j})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]\}$

(v) $[(p\mathbf{j}*\mathbf{k})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})] = p\{\operatorname{Var}[(\mathbf{j}*\mathbf{k})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]\}$

VIa. *Table* 6a

(i) $\operatorname{Var}[(\mathbf{f}*\mathbf{g}*\mathbf{h})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$= N^3\operatorname{Var}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2)+3N\left(\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right)\operatorname{Var}(2\mathbf{X}^2\mathbf{VWYZ})$$

$$+\frac{1}{40}(11N^5-40N^4+45N^3-20N^2+4N)\operatorname{Var}(2\mathbf{TVWXYZ})$$

$$+6N\binom{N}{2}^2\operatorname{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2, \mathbf{X}^2\mathbf{V}^2\mathbf{W}^2)$$

$$+12N\binom{N}{2}^2\operatorname{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2, \mathbf{X}^2\mathbf{Y}^2\mathbf{W}^2)$$

$$+\left[6\left(\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right)\binom{N}{2}\right]\operatorname{Cov}(2\mathbf{X}^2\mathbf{VWYZ}, \mathbf{T}^2\mathbf{VWYZ})$$

(ii) $\operatorname{Var}[(\mathbf{f}*\mathbf{g}*\mathbf{j})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$= N^3\operatorname{Var}(\mathbf{X}^2\mathbf{Y}^2\mathbf{VW})+4N\left[\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right]\operatorname{Var}(\mathbf{X}^2\mathbf{VWYZ})$$

$$+\frac{1}{60}(33N^5-80N^4+75N^3-40N^2+12N)\operatorname{Var}(\mathbf{TVWXYZ})$$

$$+2N\binom{N}{2}^2 \operatorname{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{TZ}, \mathbf{V}^2\mathbf{W}^2\mathbf{TZ})$$

$$+4N^2\binom{N}{2}\operatorname{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{TZ}, \mathbf{X}^2\mathbf{W}^2\mathbf{TZ})$$

$$+4\left[2\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right]\binom{N}{2}\operatorname{Cov}(\mathbf{X}^2\mathbf{VWYZ}, \mathbf{T}^2\mathbf{VWYZ})$$

(iii) $\operatorname{Var}[(\mathbf{f}*\mathbf{j}*\mathbf{k})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$=N\left(\sum_{i=1}^{N} i^2+\sum_{i=1}^{N-1} i^2\right)\operatorname{Var}(\mathbf{X}^2\mathbf{VWYZ})$$

$$+\frac{1}{60}(33N^5-40N^4+15N^3-20N^2+12N)\operatorname{Var}(\mathbf{TVWXYZ})$$

$$+2\left(\sum_{i=1}^{N} i^2+\sum_{i=1}^{N-1} i^2\right)\binom{N}{2}\operatorname{Cov}(\mathbf{X}^2\mathbf{VWYZ}, \mathbf{T}^2\mathbf{VWYZ})$$

(iv) $\operatorname{Var}[(\mathbf{j}*\mathbf{k}*\mathbf{l})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$=\left\{2\sum_{i=1}^{N}\left(\sum_{k=1}^{i} k\right)^2+2\sum_{i=1}^{(N-3)/2}\left(\sum_{j=1}^{N} j-\sum_{k=1}^{i} k+\sum_{l=N-i}^{N-1} l\right)^2\right.$$

$$\left.+\left(\sum_{j=1}^{N} j-\sum_{k=1}^{(N-1)/2} k+\sum_{l=(N+1)/2}^{N-1} l\right)^2\right\}\operatorname{Var}(\mathbf{TVWXYZ}).$$

VIb. *Table* 6b

(i) $\operatorname{Var}[(p\mathbf{f}*\mathbf{g}*\mathbf{h})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$=pN^3\operatorname{Var}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2)+\left\{pN\left[\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right]\right.$$

$$\left.+2N\left[p^2\left(\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right)\right]\right\}\operatorname{Var}(2\mathbf{X}^2\mathbf{VWYZ})$$

$$+2N\left[(1-p)-\binom{N-pN}{2}\left(\frac{1}{N}\right)^2\right]$$

$$\times\left[\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right]\operatorname{Var}(\mathbf{X}^2\mathbf{VWYZ})$$

$$+\frac{p^2}{40}(11N^5-40N^4+45N^3-20N^2+4N)\operatorname{Var}(2\mathbf{TVWXYZ})$$

$$+\frac{1-p^2-(1-p)^2}{40}(11N^5-40N^4$$

$$+45N^3-20N^2+4N)\operatorname{Var}(\mathbf{TVWXYZ})$$

$$+2\left[N^2\binom{pN}{2}+2pN^2\binom{N}{2}\right]\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2,\mathbf{X}^2\mathbf{Y}^2\mathbf{W}^2)$$

$$+2\left[pN\binom{N}{2}^2+2N\binom{N}{2}\binom{pN}{2}\right]\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2,\mathbf{X}^2\mathbf{V}^2\mathbf{W}^2)$$

$$+2\left[\binom{pN}{2}\left(\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right)\right.$$

$$\left.+2p^2\binom{N}{2}\left(\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right)\right]\mathrm{Cov}(2\mathbf{X}^2\mathbf{VWYZ},2\mathbf{T}^2\mathbf{VWYZ})$$

$$+2\left\{2\binom{N}{2}\left[(1-p)-\binom{N-pN}{2}\left(\frac{1}{N}\right)^2\right]\right.$$

$$\left.\times\left[\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right]\right\}\mathrm{Cov}(\mathbf{X}^2\mathbf{VWYZ},\mathbf{T}^2\mathbf{VWYZ})$$

(ii) $\mathrm{Var}[(p\mathbf{f}*\mathbf{g}*\mathbf{j})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$=pN^3\,\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2\mathbf{VW})$$

$$+2pN\left(\sum_{i=1}^{N}i(i-1)+\sum_{i=1}^{N-1}i(i-1)\right)\mathrm{Var}(\mathbf{X}^2\mathbf{VWYZ})$$

$$+\frac{p}{60}(33N^5-80N^4+75N^3-40N^2+12N)\,\mathrm{Var}(\mathbf{TVWXYZ})$$

$$+2\left[N\binom{N}{2}\binom{pN}{2}\right]\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{TZ},\mathbf{V}^2\mathbf{W}^2\mathbf{TZ})$$

$$+2\left[pN^2\binom{N}{2}+N^2\binom{pN}{2}\right]\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{TZ},\mathbf{X}^2\mathbf{W}^2\mathbf{TZ})$$

$$+2\left\{\binom{N}{2}2p\left[\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right]\right.$$

$$\left.+\binom{pN}{2}2\left[\sum_{i=1}^{N}\binom{i}{2}+\sum_{i=1}^{N-1}\binom{i}{2}\right]\right\}\mathrm{Cov}(\mathbf{X}^2\mathbf{VWYZ},\mathbf{T}^2\mathbf{VWYZ})$$

(iii) $\mathrm{Var}[(\mathbf{f}*\mathbf{g}*p\mathbf{j})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]=p\{\mathrm{Var}[(\mathbf{f}*\mathbf{g}*\mathbf{j})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]\}$

(iv) $\mathrm{Var}[p\mathbf{f}*\mathbf{j}*\mathbf{k})\cdot(\mathbf{f}*\mathbf{g}*\mathbf{h})]$

$$=pN\left(\sum_{i=1}^{N}i^2+\sum_{i=1}^{N-1}i^2\right)\mathrm{Var}(\mathbf{X}^2\mathbf{VWYZ})$$

$$+\frac{p}{60}(33N^5-40N^4+15N^3-20N^2+12N)\,\mathrm{Var}(\mathbf{TVWXYZ})$$

$$+2\left(\sum_{i=1}^{N}i^2+\sum_{i=1}^{N-1}i^2\right)\binom{pN}{2}\mathrm{Cov}(\mathbf{X}^2\mathbf{VWYZ},\mathbf{T}^2\mathbf{VWYZ})$$

(v) $\mathrm{Var}[(\mathbf{f} * \mathbf{j} * p\mathbf{k}) \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})] = p\{\mathrm{Var}\{(\mathbf{f} * \mathbf{j} * \mathbf{k}) \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})]\}$

(vi) $\mathrm{Var}[(p\mathbf{j} * \mathbf{k} * \mathbf{l}) \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})] = p\{\mathrm{Var}\{(\mathbf{j} * \mathbf{k} * \mathbf{l}) \cdot (\mathbf{f} * \mathbf{g} * \mathbf{h})]\}$.

APPENDIX B

Variance and Covariance Values of Combinations of Random Variables Occurring in Appendix A

For $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{S}, \mathbf{T}, \mathbf{V}, \mathbf{W}$ independent and identically distributed as $N(0, \sigma^2)$,

$$E(\mathbf{X}^2) = \sigma^2$$
$$E(\mathbf{X}^4) = 3\sigma^4$$
$$E(\mathbf{X}^6) = 15\sigma^6$$
$$E(\mathbf{X}^8) = 105\sigma^8$$

$\mathrm{Var}(\mathbf{X}^2) = 2\sigma^4$

$\mathrm{Var}(\mathbf{XY}) = \sigma^4$

$\mathrm{Var}(\mathbf{X}^3) = 15\sigma^6$

$\mathrm{Var}(\mathbf{X}^2\mathbf{Y}) = 3\sigma^6$

$\mathrm{Var}(\mathbf{XYZ}) = \sigma^6$

$\mathrm{Var}(\mathbf{X}^3\mathbf{YZ}) = 15\sigma^{10}$

$\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}) = 8\sigma^{10}$

$\mathrm{Var}(\mathbf{X}^2\mathbf{WYZ}) = 2\sigma^{10}$

$\mathrm{Var}(\mathbf{VWXYZ}) = \sigma^{10}$

$\mathrm{Cov}(\mathbf{Z}^3, \mathbf{X}^2\mathbf{Z}) = 3\sigma^6$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}, \mathbf{Z}^2\mathbf{Y}) = \sigma^8$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}, \mathbf{X}^2\mathbf{W}^2\mathbf{Z}) = 3\sigma^{10}$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{W}, \mathbf{T}^2\mathbf{V}^2\mathbf{W}) = \sigma^{10}$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{WYZ}, \mathbf{V}^2\mathbf{WYZ}) = \sigma^{10}$

$\mathrm{Var}(\mathbf{X}^4) = 96\sigma^8$

$\mathrm{Var}(\mathbf{X}^3\mathbf{Y}) = 15\sigma^8$

$\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2) = 8\sigma^8$

$\mathrm{Var}(\mathbf{X}^2\mathbf{YZ}) = 3\sigma^8$

$\mathrm{Var}(\mathbf{WXYZ}) = \sigma^8$

$\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2) = 26\sigma^{12}$

$\mathrm{Var}(\mathbf{X}^2\mathbf{Y}^2\mathbf{VW}) = 9\sigma^{12}$

$\mathrm{Var}(\mathbf{X}^2\mathbf{VWYZ}) = 3\sigma^{12}$

$\mathrm{Var}(\mathbf{TVWXYZ}) = \sigma^{12}$

$\mathrm{Cov}(\mathbf{X}^4, \mathbf{X}^2\mathbf{Y}^2) = 12\sigma^8$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2, \mathbf{X}^2\mathbf{Z}^2) = 2\sigma^8$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{YZ}, \mathbf{W}^2\mathbf{YZ}) = \sigma^8$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2, \mathbf{X}^2\mathbf{Y}^2\mathbf{V}^2) = 8\sigma^{12}$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{VW}, \mathbf{X}^2\mathbf{T}^2\mathbf{VW}) = 3\sigma^{12}$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{Z}^2, \mathbf{X}^2\mathbf{V}^2\mathbf{W}^2) = 2\sigma^{12}$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{Y}^2\mathbf{TS}, \mathbf{V}^2\mathbf{W}^2\mathbf{TS}) = \sigma^{12}$

$\mathrm{Cov}(\mathbf{X}^2\mathbf{VWYZ}, \mathbf{T}^2\mathbf{VWYZ}) = \sigma^{12}$

APPENDIX C

Expected values and variances of resemblance distributions of probe types and memory vector for fit of Clark and Shiffrin (1987) data. Code: $ABC = 1$, $ABC' = 2$, $AB'C'' = 3$, $ABX = 4$, $AB'X = 5$, $AXY = 6$, $XYZ = 7$, $AB = 8$, $AB' = 9$, $AX = 10$, $XY = 11$, $A = 12$, $X = 13$. Expressions Q are component variances used to compute VAR(1) to VAR(13). The arguments of Q do not refer to the type of probe.

$$EV(1) = c_j(t\psi + 2d\omega + 3s\gamma)$$

$$EV(2) = EV(3) = EV(4) = EV(5) = EV(6) = EV(7) = 0$$

$$EV(8) = c_j(p\omega + 2q\gamma)$$

$$EV(9) = EV(10) = EV(11) = 0$$

$$EV(12) = c_j w\gamma$$

$$EV(13) = 0$$

$$Q(1) = (5N^3 + 3N^2 + N)/3N^4$$

$$Q(2) = (2N^3 + N)/3N^4$$

$$Q(3) = (595N^4 + 384N^3 + 722N^2 - 192N + 27)/192N^5$$

$$Q(4) = (259N^4 + 98N^2 + 27)/192N^5$$

$$Q(5) = (115N^4 + 50N^2 + 27)/192N^5$$

$$Q(6) = (73N^5 + 40N^4 + 35N^3 + 20N^2 + 12N)/60N^6$$

$$Q(7) = (33N^5 + 15N^3 + 12N)/60N^6$$

$$Q(8) = (7N^2 + 4N + 1)/4N^3$$

$$Q(9) = (3N^2 + 1)/4N^3$$

$$Q(10) = (16N^3 + 6N^2 + 2N)/3N^4$$

$$Q(11) = 1/N$$

$$Q(12) = 2/N$$

$$Q(13) = (143N^5 + 260N^4 + 85N^3 + 40N^2 + 12N)/60N^6$$

$$Q(14) = (486N^5 + 960N^4 - 30N^3 + 120N^2 + 24N)/60N^6$$

$$\begin{aligned}\mathrm{VAR}(1) = {} & Q(1)\, c_j(3t\gamma + 2d\omega + 2s\psi) + Q(2)k(3t\gamma + 4d\omega + 3s\psi) \\ & + Q(3)\, c_j(2t\omega + 2d\psi) + Q(14)\, c_j t\psi + Q(5)\, k(2t\omega + 2d\psi) \\ & + Q(7)\, kt\psi + Q(8)\, c_j(4d\gamma + 4s\omega) \\ & + Q(9)(2c_j s\omega + 6ks\omega + 2d\gamma(c_j + k)) + Q(10)\, 2c_j\, d\omega \\ & + Q(11)\, 6c_j\gamma + Q(12)\, 3c_j s\gamma\end{aligned}$$

$$
\begin{aligned}
\mathrm{VAR}(2) = {} & Q(1)(c_j(2t\gamma + 3d\omega + 2s\psi) + c_l(t\gamma + d\omega + s\psi)) \\
& + Q(2)(k'(3t\gamma + 3s\psi + 4d\omega) + c_j(t\gamma + s\psi) + c_l(2t\gamma + 3d\omega + 2s\psi)) \\
& + Q(3)\, c_j(t\omega + d\psi) + Q(4)(t\omega + d\psi)(c_j + c_l) + Q(13)\, c_j t\psi \\
& + Q(5)(c_l + 2k')(t\omega + d\psi) + Q(6)\, c_l t\psi + Q(7)\, k' t\psi \\
& + Q(8)(3c_j + c_l)(d\gamma + s\omega) + Q(9)(3c_j + 5c_l + 6k')(d\gamma + s\omega) \\
& + Q(10)\, c_j d\omega + Q(11)(7c_j + 8c_l + 9k')\, s\gamma + Q(12)(2c_j + c_l)\, s\gamma
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{VAR}(3) = {} & Q(1)(c_j + c_l + c_m)(t\gamma + s\psi) + d\omega(c_j + c_m + 4c_l) \\
& + Q(2)((c_j + c_l + c_m)\, 2t\gamma + k''(3t\gamma + 2d + 3s) \\
& + 3d\omega(c_j + c_m) + 2s\psi(c_l + c_m)) \\
& + Q(4)((c_j + c_l + c_m)\, t\omega + (c_j + c_m + 2c_l)\, d\psi) \\
& + Q(5)(t\omega(2k'' + c_j + c_l + c_m) + d\psi(2k'' + c_j + c_m)) \\
& + Q(6)(c_j + c_l + c_m)\, t\psi + Q(7)\, k'' t\psi + Q(8)(c_j + 2c_l + c_m)(d\gamma + s\omega) \\
& + Q(9)((d\gamma + s\omega)(c_j + 4c_l + 5c_m + 6k'') + 4d\gamma c_j) \\
& + Q(11)\, s\gamma(2c_j + 8c_l + 8c_m + 9k'') + Q(12)\, s\gamma(c_j + c_l + c_m)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(4) = {} & Q(1)\, c_j(2t\gamma + 3d\omega + 2s\psi) + Q(2)((c_j + 3k)(t\gamma + s\psi) + 4kd\omega) \\
& + Q(4)\, c_j(t\omega + d\psi) + Q(3)\, c_j(t\omega + d\psi) + Q(13)\, c_j t\psi \\
& + Q(5)\, 2k(t\omega + d\psi) + Q(7)\, kt\psi + Q(8)\, 3c_j(d\gamma + s\omega) \\
& + Q(9)(3c_j + 6k)(d\gamma + s\omega) \\
& + Q(10)\, c_j d\omega + Q(11)(7c_j + 9k)\, s\gamma + Q(12)\, 2c_j s\gamma
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{VAR}(5) = {} & Q(1)((c_j + c_l)(t\gamma + d\omega + s\psi) + 3c_l\, d\omega) \\
& + Q(2)((c_j + c_l)\, 2t\gamma + 3k' t\gamma + (3c_j + 4k')\, d\omega + (2c_j + 2c_l + 3k')\, s\psi \\
& + Q(4)(c_j + 2c_l)(t\omega + d\psi) + Q(5)(c_j + 2k')(t\omega + d\psi) \\
& + Q(6)(c_j + c_l)\, t\psi + Q(7)\, kt\psi + Q(8)(c_j + 2c_l)(d\gamma + s\omega) \\
& + Q(9)((4c_j + 4c_l + 6k')(d\gamma + s\omega) + c_j d\gamma) \\
& + Q(11)(8c_j + 8c_l + 9k')\, s\gamma + Q(12)(c_j + c_l)\, s\gamma
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{VAR}(6) = {} & Q(1)\, c_j(t\gamma + d\omega + s\psi) \\
& + Q(2)((2c_j + 3k)\, t\gamma + (3c_j + 4k)\, d\omega + (2c_j + 3k)\, s\psi) \\
& + Q(4)\, c_j t\omega + Q(5)(c_j + 2k)(t\omega + d\psi) \\
& + Q(6)\, c_j(t + d)\, \psi + Q(7)\, kt\psi + Q(8)\, c_j(d\gamma + s\omega) \\
& + Q(9)(5c_j + 6k)(d\gamma + s\omega) + Q(11)(8c_j + 9k')\, s\gamma + Q(12)\, c_j s\gamma
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{VAR}(7) = {} & Q(2)(c_j + k)(3t\gamma + 4d\omega + 3s\psi) + Q(5)\, 2(c_j + k)(t\omega + d\psi) \\
& + Q(7)(c_j + k)\, t\psi + Q(9)\, 6(c_j + k)(d\gamma + s\omega) + Q(11)\, 9(c_j + k)\, s\gamma
\end{aligned}
$$

$$\begin{aligned}
\mathrm{VAR}(8) = {} & Q(1)\, c_j(p\omega + 2q\psi) + Q(2)\, 2k(p\omega + q\psi) + Q(3)\, c_j p\psi \\
& + Q(5)\, kp\psi + Q(8)\, c_j(2p\gamma + 3q\omega) \\
& + Q(9)((c_j + 3k)(p\gamma + q\omega) + kq\omega) \\
& + Q(10)\, c_j p\omega + Q(11)(4c_j + 6k)\, q\gamma + Q(12)\, 2c_j q\gamma
\end{aligned}$$

$$\begin{aligned}
\mathrm{VAR}(9) = {} & Q(1)(c_j(p\omega + q\psi) + 2c_l p\omega) \\
& + Q(2)((c_j + 2k')(p\omega + q\psi) + 2c_l q\psi) \\
& + Q(4)(c_j + c_l)\, p\psi + Q(5)\, k'p\psi + Q(8)((c_j + c_l)(p\gamma + q\omega) + c_l q\omega) \\
& + Q(9)((2c_j + 2c_l + 3k')(p\gamma + q\omega) + (c_j + k')\, q\omega) \\
& + Q(11)(5c_j + 5c_l + 6k')\, q\gamma + Q(12)(c_j + c_l)\, q\gamma
\end{aligned}$$

$$\begin{aligned}
\mathrm{VAR}(10) = {} & Q(1)\, c_j(p\omega + q\psi) + Q(2)(c_j + 2k)(p\omega + q\psi) \\
& + Q(4)(c_j p\psi) + Q(5)\, kp\psi + Q(8)\, c_j(p\gamma + q\omega) \\
& + Q(9)(c_j(2p\gamma + 3q\omega) + k(3p\gamma + 4q\omega)) \\
& + Q(11)(5c_j + 6k)\, q\gamma + Q(12)\, c_j q\gamma
\end{aligned}$$

$$\begin{aligned}
\mathrm{VAR}(11) = {} & Q(2)\, 2(c_j + k)(p\omega + q\psi) + Q(5)(c_j + k)\, p\psi \\
& + Q(9)(c_j + k)(3p\gamma + 4q\omega) + Q(11)\, 6(c_j + k)\, q\gamma
\end{aligned}$$

$$\begin{aligned}
\mathrm{VAR}(12) = {} & Q(1)\, c_j w\psi + Q(2)\, kw\psi + Q(8)\, c_j w\omega + Q(9)(c_j + 2k)\, w\omega \\
& + Q(11)(2c_j + 3k)\, w\gamma + Q(12)\, c_j w\gamma
\end{aligned}$$

$$\mathrm{VAR}(13) = Q(2)(c_j + k)\, w\psi + Q(9)\, 2(c_j + k)\, w\omega + Q(11)\, 3(c_j + k)\, w\gamma$$

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